Theory of Rings Homework

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Rings3p1:

An ideal $P \subseteq R$ is *prime* if whenever I and J are ideals satisfying $IJ \subseteq P$, then $I \subseteq P$ or $J \subseteq P$. The intersection of all prime ideals is called the *prime radical* of R. Show that the prime radical of a ring is a nil ideal, hence is contained in $\text{rad}(R)$. (Hint: show that if $r \in R$ is not nilpotent, then there is a prime ideal of R not containing r.

Proof: Let $r \in R$ be a not nilpotent element. Thus, $r^k \neq 0$ for all $k \in \mathbb{N}$. In order to make a prime ideal, we'll try to find a sort of maximal ideal. Consider the set $A = \{I : I \leq R,\ r^k \notin I \text{ for all } k \in \mathbb{N}\}$. This is not an empty because (0) is an ideal of R and $r^k \notin (0)$ because r is not nilpotent. Now, we can partially order A by inclusion. Consider one totally ordered subset of A, $\{I_\alpha\}$. Then, $\cup I_\alpha$ is an ideal of R and $r^k \notin \cup I_\alpha$. Thus, each totally ordered chain of ideals in A will have an upper bound in A. Now, we can apply Zorn's Lemma. A will contain a maximal ideal, say P. Moreover, $r^k \notin P$ by construction. Claim: P is a prime ideal. Let $x, y \in R$ and $x \notin P$ and $y \notin P$. Let X be an ideal containing x and Y be an ideal containing y. $X \not\subseteq P$ and $Y \not\subseteq P$. Then $X + P \not\subseteq P$ and $Y + P \not\subseteq P$. In particular, $r^m = x_m + p_m$ for some $x_m \in X$ and $p_m \in P$, and $r^n = y_n + p_n$ for some $y_n \in Y$ and $p_n \in P$. Now, $r^{m+n} = r^m r^n = (x_m + p_m)(y_n + p_n) = (x_m y_n + x_m p_n + p_m y_n + p_m p_n) \in XY + P$ because $x_m \in X$ and $y_n \in Y$ and $p_m, p_n \in P$. Thus, $XY + P \not\subseteq P$, which means that $XY \not\subseteq P$. Thus, P is a prime ideal. Thus the prime radical contains only nilpotent elements. And because the intersection of any number of ideals is an ideal, the prime radical is a nil ideal. Finally, Lemma 4.11 gives that the prime radical is contained in $\text{rad}(R)$.