Problem. Let V be a finite dimensional complex vector space. A linear transformation T in $\mathrm{End}_{\mathbb{C}}(V)$ is called *semisimple* if it "acts semisimply on V", by which we mean that V is a semisimple module under the $\mathbb{C}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(V)$ that is generated by T. Show that T is semisimple iff it is diagonalizable.

Solution. ($\Rightarrow$) Suppose T is semisimple. Then V is a semisimple module under the $\mathbb{C}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(V)$ that is generated by T. We'll refer to this subalgebra as A.

So $V = V_1 \oplus \cdots \oplus V_n$ where each V_i is a simple A-module. Since each V_i is a simple A-module, it must be closed under the action of elements of A, hence under the action of T. So for any $v \in V_i$, we have that $T(v) \in V_i$ also.

Moreover, since each V_i is simple as an A-module, each V_i has no nontrivial A-submodules.

Now suppose that some V_i is not a one-dimensional $\mathbb{C}$-vector space. For any nonzero $v \in V_i$, if $T(v) = \lambda v$ for some $\lambda \in \mathbb{C}$, then $\langle v \rangle$ is a nontrivial proper A-submodule of V_i, giving a contradiction. So $T(v) \neq \lambda v$ for all $\lambda \in \mathbb{C}$ and nonzero $v \in V_i$; in other words, V_i contains no eigenvectors of T.

Since V is a finite-dimensional vector space, so is V_i, so we can choose a basis for V_i. Since T maps V_i into V_i, the matrix of $T|_{V_i}$ with respect to this basis is square. Performing row reduction on this matrix produces at least one eigenvector of T in V_i, giving a contradiction.

Therefore each V_i is a one-dimensional $\mathbb{C}$-vector space as well as being a simple A-module. So $V = V_1 \oplus \cdots \oplus V_n$ as a $\mathbb{C}$-vector space, with $T(V_i) = V_i$. Letting each $V_i = \langle v_i \rangle$ for some $v_i \in V$, we have that $T(v_i) = \lambda_i v_i$ for some $\lambda_i \in \mathbb{C}$ and $\{v_1, \ldots, v_n\}$ is a basis for V. So T is diagonalizable with respect to this basis.

($\Leftarrow$) Suppose that T is diagonalizable. Then there exists a basis $\{v_1, \ldots, v_n\}$ for V such that $T(v_i) = \lambda_i v_i$ for some $\lambda_i \in \mathbb{C}$. Now

$$V = \bigoplus_{i=1}^{n} \langle v_i \rangle$$

as a $\mathbb{C}$-vector space and hence

$$V = \sum_{i=1}^{n} \langle v_i \rangle$$

as an A-module. Also, each $\langle v_i \rangle$ is a simple A-module since it is a simple $\mathbb{C}$-module and $T(v_i) \in \langle v_i \rangle$. By Theorem 2.4, V is a semisimple A-module since V is a sum of simple A-modules.