MATH 6250: Theory of Rings
Homework 2
E. Shannon, K. Havasi

Ex. 3.6A. Let M be a left R-module and $E = End({}_RM)$. If ${}_RM$ is a semisimple R-module, show that M_E is a semisimple E-module.

Ex. 3.6B. In the above Exercise, if M_E is a semisimple E-module, is ${}_RM$ necessarily a semisimple R-module?

Solution.

(A) Suppose ${}_RM$ is a semisimple R-module. Then it is the direct sum of simple R modules, so ${}_RM = \bigoplus_{i\in I} {}_RM^i$, where the R-submodules ${}_RM^i$ are simple. This means that elements of ${}_RM^i$ generate ${}_RM$ using addition only. Let's denote by M_E^i the same set of elements as ${}_RM^i$, but now considered as a subset of the E-module M_E. In M_E the subsets M_E^i are not necessarily submodules, as they may not be closed under E-multiplications. As the additive structure is the same in ${}_RM$ and in M_E, the subsets M_E^i of the E-module M_E will generate M_E using additions only, so $M_E = \sum_{i\in I} M_E^i$. Now let's regroup the subsets M_E^i of the E-module M_E according to their isomorphism types as R-modules ${}_RM^i$ and write $M_E = \sum_{j\in J}\sum_{k\in K_j} M_E^k$. Here $I = \bigcup_{j\in J} K_j$ and for any $j \in J$ and $k_1, k_2 \in K_j$: ${}_RM^{k_1} \cong {}_RM^{k_2}$ as R-modules. Our claim is that the outside sum (going over the index set J) is a direct sum. We know that it is a direct sum if we only consider the additive group, so we only need to check that $\sum_{k\in K_j} M_E^k$ is in fact a submodule of M_E, that is it is closed under multiplication from E. If this was not the case, then there would be a homomorphism $\varphi \in E$ and some element $m \in {}_RM^k, k \in K_j$, such that $\varphi(m)$ would have a non-zero projection to some ${}_RM^{k'}, k' \notin K_j$, and thus we could create a non-zero homomorphism between two non-isomorphic simple R-modules. This is impossible, therefore we have that $M_E = \bigoplus_{j\in J}\sum_{k\in K_j} M_E^k$, and we only need to prove that $\sum_{k\in K_j} M_E^k$ is semisimple as E-module for any $j \in J$. From now on we fix a $j \in J$ and assume that all the ${}_RM^k$'s are isomorphic as R-modules.
From Schur's Lemma we know that any ${}_RM^k$ is a vector space over the ring of endomorphisms of ${}_RM^k$. Thus we can write the additive group of ${}_RM^k$ as a direct sum of one dimensional subspaces, so $M^k = \bigoplus_{s\in S}\langle e_s^k\rangle$, where M^k denotes the additive group of ${}_RM^k$ equipped with the vector space structure over the division ring of endomorphisms of ${}_RM^k$. The ring of endomorphisms of ${}_RM^k$ acts on M^k as scalar multiplication. If we pick an arbitrary endomorphism $\varphi \in E$, restrict it to some ${}_RM^{k_1}$ and take its projection to some ${}_RM^{k_2}$, then we must have an endomorphism of ${}_RM^k$, using that now ${}_RM^{k_1} \cong {}_RM^{k_2}$. This means that if we form the additive groups $N_s = \bigoplus_{k\in K}\langle e_s^k\rangle$ for all $s \in S$, then this group N_s is also a submodule of M_E, because it is closed under multiplications from E, as they act on any ${}_RM^k$ as scalar multiplication so cannot take any $e_{s_1}^{k_1}$ to another $e_{s_2}^{k_2}$ if $s_1 \neq s_2$. Furthermore it is simple as E-module, because any

non-zero element of N_s can be taken to any other element of N_s by E-multiplication. To see this, first we prove that any non-zero element can be taken to some e_s^k. This is true, as the endomorphisms of ${}_RM$ contain the projections to any direct summand ${}_RM^k$, furthermore the endomorphism ring of ${}_RM$ contains the division ring of endomorphisms of ${}_RM^k$ which act like scalar multiplication on the elements of $\langle e_s^k \rangle$. Now any element $e_s^{k_1}$ can be taken to any other element $e_s^{k_2}$ by an endomorphism of ${}_RM$, because ${}_RM^{k_1}$ and ${}_RM^{k_2}$ are direct summands of ${}_RM$, and they are isomorphic by an isomorphism that takes $e_s^{k_1}$ to $e_s^{k_2}$. As the endomorphism ring of ${}_RM$ is closed under addition, we can take any element e_s^k to any element of N_s. Thus indeed we can take any non-zero element of N_s into any element of N_s by E-multiplications. As $\sum_{k \in K_j} M_E^k = \bigoplus_{s \in S} N_s$ and the N_s are simple, we have that $\sum_{k \in K_j} M_E^k$ is semisimple and thus M_E is semisimple.

(B) Let's consider the module ${}_{\mathbb{Z}}\mathbb{Q}$. Modules over $\mathbb{Z}$ are the abelian groups, and simple modules over $\mathbb{Z}$ are the simple abelian groups, that is the prime order groups. As ${}_{\mathbb{Z}}\mathbb{Q}$ is not the sum of prime order groups, it is not semisimple. The endomorphism ring of ${}_{\mathbb{Z}}\mathbb{Q}$ contains all the multiplications by rational numbers, which means that if $E = End({}_{\mathbb{Z}}\mathbb{Q})$, then $\mathbb{Q}$ is a subring of E. But then in $\mathbb{Q}_E$ we can take any non-zero element to any element by multiplication and therefore $\mathbb{Q}_E$ must be simple and thus semisimple. We have found a counterexample and therefore if M_E is a semisimple E-module, ${}_RM$ is not necessarily semisimple as R-module.