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Let V be an infinite dimensional D-vector space.

(a) Show that if I is a 2-sided ideal of $R = \mathrm{End}_D(V)$ and $r \in I - \{0\}$ has infinite rank λ, then I contains every element of r whose rank is $\leq \lambda$.

Let $s \in R$ have rank $\leq \lambda$; then $\dim(\ker(s)) \geq \dim(\ker(r))$, and there exists an isomorphism $f \in R$ such that $\ker(r) \subseteq f(\ker(s))$. Note that $\ker(rf) \subseteq \ker(s)$, and in particular if $s(v_1) \neq s(v_2)$ then $rf(v_1) \neq rf(v_2)$.

Define $g : V \to V$ by extending the endomorphism defined by

$$\begin{aligned} g : \mathrm{im}(rf) &\to V \\ rf(v) &\mapsto s(v) \end{aligned}$$

to all of V (Note: g is well-defined by the above remark and an endomorphism as s and rf are both endomorphisms). Then $s = grf$, and in particular $s \in I$.

(b) Show that the two-sided ideals of R are of the two types

$$\begin{aligned} I_{<\lambda} &= \{e \in R \mid \mathrm{rank}(e) < \lambda\} \\ I_{\leq\lambda} &= \{e \in R \mid \mathrm{rank}(e) \leq \lambda\} \end{aligned}$$

for some infinite cardinal $\lambda \leq \mathrm{Dim}_D(V)$.

Lemma: Let I be a two-sided ideal of R, and let $r \in I - \{0\}$ be an element of finite rank n. Then I contains all elements of R of finite rank.

Proof. Note that the proof of part (a) does not rely on r having infinite rank; it suffices to show that I contains some element of rank $> n$, and the result will follow by induction.

Let $\{r(v_1), ..., r(v_n)\}$ be a basis of $\mathrm{im}(r)$, and let v_{n+1} and w be elements of V such that $\{v_1, ..., v_{n+1}\}$ and $\{r(v_1), ..., r(v_n), w\}$ are linearly independent sets and $r(v_{n+1}) = 0$. Let $f \in R$ be an endomorphism such that

$$f(v_i) = \begin{cases} v_1 & \text{if } i = n+1 \\ 0 & \text{else} \end{cases}$$

and let $g \in R$ be an endomorphism such that $gr(v_1) = w$.

Consider $h = r + grf$; as $h(v_i) = r(v_i)$ for $1 \leq i \leq n$ and $h(v_{n+1}) = w$, it is clear that h has rank greater than n. It follows that h is the desired element of I.

□

Now let I be an ideal of R, and let

$$\lambda = \sup_{r \in I}(\mathrm{rank}(r))$$

By the lemma, λ is an infinite cardinal. If there exists $r \in I$ with $\mathrm{rank}(r) = \lambda$, then I contains all elements of R with rank $\leq \lambda$ by part (a). By the definition of λ, no elements of R of rank $> \lambda$ are contained in I, hence I is of the form $I_{\leq\lambda}$.

Now assume there is no $r \in I$ with $\mathrm{rank}(r) = \lambda$; let $r \in R$ with $\mathrm{rank}(r) = \mu < \lambda$. By definition of λ, there exists $s \in I$ with $\mathrm{rank}(s) > \mu$, hence by part (a) $r \in I$. It follows that I is of the form $I_{<\lambda}$.