**Problem.** (Restriction of scalars.) Let $\varphi : R \to S$ be a ring homomorphism.

(i) If $\rho : S \to \mathrm{End}_{\mathbb{Z}}(M)$ is the structure map for an $S$-module, then the composite $\rho \circ \varphi : R \to S \to \mathrm{End}_{\mathbb{Z}}(M)$ is the structure map for an $R$-module. This defines an assignment ${}_SM \to {}_RM$. Show that this is the object part of a functor from ${}_SMod$ to ${}_RMod$ (called 'restriction of scalars').

(ii) Since restriction of scalars is a functor, it takes isomorphic $S$-modules to isomorphic $R$-modules. Show that if $\varphi$ is an epimorphism of rings, then restriction of scalars takes nonisomorphic $S$-modules to nonisomorphic $R$-modules.

**Solution.**

(i) Fixing some ring homomorphism $\varphi : R \to S$, we'll define the relevant functor. Define $F : {}_SMod \to {}_RMod$ by:

- $F$ identifies with any left $S$-module ${}_SM$ a left $R$-module ${}_RM$ in the following way: elements of ${}_RM$ are just the elements of ${}_SM$, with the same addition, and $R$ acts on ${}_RM$ by $r \cdot m = \varphi(r)m$. This makes sense because $\varphi(r)$ is an element of $S$ and we are given an action of $S$ on the elements of ${}_SM$ (the same elements).
- For each $S$-module homomorphism $\phi : {}_SA \to {}_SB$, $F$ assigns an $R$-module homomorphism $F(\phi) : {}_RA \to {}_RB$ given by, for $a \in {}_RA$, $F(\phi)(a) = \phi(a)$ where we view $a$ temporarily as an element of ${}_SA$, apply $\phi$, then view the result again as an element of ${}_RA$.

  This $F(\phi)$ truly is an $R$-module homomorphism ${}_RA \to {}_RB$, since
  $F(\phi)(a_1 + a_2) = \phi(a_1 + a_2) = \phi(a_1) + \phi(a_2) = F(\phi)(a_1) + F(\phi)(a_2)$, and
  $F(\phi)(r \cdot a) = \phi(r \cdot a) = \phi(\varphi(r)a) = \varphi(r)\phi(a) = r \cdot \phi(a) = r \cdot F(\phi)(a)$.
  Moreover (more or less by definition) we have that if $\phi_1 : {}_SB \to {}_SC$ and $\phi_2 : {}_SA \to {}_SB$, then $F(\phi_1 \circ \phi_2) = F(\phi_1) \circ F(\phi_2)$.

The assignment described (${}_SM \to {}_RM$) in the problem is the object part of this functor.

(ii) In the case where $\varphi$ is surjective, it's easy to show that $F$ takes non isomorphic $S$-modules to non isomorphic $R$-modules.

Suppose $f$ is an isomorphism between two $R$-modules ${}_RA$ and ${}_RB$ where $F({}_SA) = {}_RA$ and $F({}_SB) = {}_RB$ for some ${}_SA, {}_SB \in {}_SMod$. Then $f$ is a bijective $R$-module homomorphism ${}_RA \to {}_RB$. Since ${}_RA$ has the same elements as ${}_SA$, and ${}_RB$ has the same elements as ${}_SB$, define a new mapping $g : {}_SA \to {}_SB$ on the level of sets by $g(a) = f(a)$, viewed as an element of ${}_SB$.

Clearly $g$ is a bijection since the sets have not changed. Also, $g(a_1 + a_2) = g(a_1) + g(a_2)$ as before (for $a_a, a_2 \in {}_SA$ now) since $f$ was an $R$-module homomorphism on these same elements. I claim that $g$ is also an $S$-module homomorphism. Let $s \in S$. Since $\varphi$ is assumed to be surjective, let $r \in R$ such that $\varphi(r) = s$. Since $f$ was an $R$-module homomorphism, and by the definition of $R$ acting on ${}_RA$, we have

$$g(sa) = f(sa) = f(\varphi(r)a) = f(r \cdot a) = r \cdot f(a) = \varphi(r)f(a) = sf(a) = sg(a).$$

So $g$ is a bijective $S$-module homomorphism ${}_SA \to {}_SB$, so ${}_SA$ and ${}_SB$ are isomorphic as $S$-modules. Therefore $F$ takes non-isomorphic $S$-modules to non-isomorphic $R$-modules.

We now address the case where $\varphi$ may not be surjective, but is still an epimorphism.

Suppose $M_1$ and $M_2$ are non isomorphic $S$-modules, but that $M_1$ and $M_2$ are isomorphic as $R$-modules under our construction.

Since $M_1$ and $M_2$ are isomorphic as $R$-modules, there is clearly a bijection between elements of $M_1$ and elements of $M_2$. Moreover, this bijection is a homomorphism with respect to addition, so $M_1$ and $M_2$ are isomorphic as abelian groups. Since the addition does not change when $M_1$ and $M_2$ are viewed as $R$-modules versus $S$-modules, this means we might as well assume that we have one module, $M$, with two different $S$-module structures on $M$, ie. two different structure maps $\rho_1$ and $\rho_2$ from $S$ to $\mathrm{End}_{\mathbb{Z}}(M)$ – but that $\rho_1$ and $\rho_2$ induce the same $R$-module structure on $M$.

Therefore $\rho_1 \circ \varphi = \rho_2 \circ \varphi$ when viewed as maps from $R$ to $\mathrm{End}_{\mathbb{Z}}(M)$. But $\varphi$ is an epimorphism, so it must be that $\rho_1 = \rho_2$ in the first place. So when $\varphi$ is an epimorphism, $F$ takes non isomorphic $S$-modules to non isomorphic $R$-modules.