Problem. (The localization functor.) An element $e \in R$ is idempotent if it satisfies $e^2 = e$. If e is an idempotent of R, then $eRe = \{ere \,|\, r \in R\}$ is a ring under the operations $\cdot$, $+$, $-$, 0 and e (as multiplicative identity element). Moreover, if M is an R-module, then $eM = \{em \,|\, m \in M\}$ is an eRe-module under the operations inherited from M. If $\varphi : M \to N$ is an R-module homomorphism, then the restriction $\varphi|_{eM} : eM \to eN$ is an eRe-module homomorphism.

(i) Show that $M \to eM$, $\varphi \mapsto \varphi|_{eM}$ is a functor from ${}_R Mod$ to ${}_{eRe}Mod$.

(ii) Show that this functor maps a simple R-module either to a zero eRe-module or to a simple eRe-module.

(iii) Show that if the ring R is semisimple, then so is eRe.

Solution.

(i) Let $F : {}_R Mod \to {}_{eRe}Mod$ be defined on objects by $F(M) = eM$ and on morphisms by $F(\varphi) = \varphi|_{eM}$. We already know that these maps are well-defined and end up in the right places. It remains to show that F satisfies the four properties of a (covariant) functor.

(1) $F(f \circ g) = F(f) \circ F(g)$.
Let $g : A \to B$ and $f : B \to C$ be R-module homomorphisms for some R-modules A, B, and C. Then $f \circ g$ is an R-module homomorphism $A \to C$, and $F(f \circ g)$ is given by

$$F(f \circ g) : eA \to eC \qquad\qquad F(f \circ g)(x) = (f \circ g)(x) \text{ for } x \in eA$$

On the other hand, $F(g)$ maps eA to eB according to the rule

$$F(g)(x) = g(x) \text{ for } x \in eA,$$

and $F(f)$ maps eB to eC according to the rule

$$F(f)(x) = f(x) \text{ for } x \in eB.$$

So $F(f) \circ F(g)$ maps eA to eC according to the rule

$$(F(f) \circ F(g))(x) = F(f)(F(g)(x)) = F(f)(g(x)) = f(g(x)) \text{ for } x \in eA.$$

(2) $F(id_A) = id_{F(A)}$.
For $A \in {}_R Mod$, id_A maps A to A by the rule $x \mapsto x$. So $F(\mathrm{id}_A)$ maps eA to eA by the rule $x \mapsto x$ for $x \in eA$. Luckily this map is exactly id_{eA} and $eA = F(A)$.

(3) $F(domf) = domF(f)$.
Let f be a morphism in ${}_R Mod$ with $\mathrm{dom} f = A$ and $\mathrm{cod} f = B$. Then $F(\mathrm{dom} f) = F(A) = eA$. We defined $F(f)$ as a restriction of f to the new domain eA, so this works out.

(4) $F(codf) = codF(f)$.
Let f be a morphism in ${}_R Mod$ with $\mathrm{dom} f = A$ and $\mathrm{cod} f = B$. Then $F(\mathrm{cod} f) = F(B) = eB$. We defined $F(f)$ as morphism from eA to eB, so this works out.

(ii) Suppose M is a simple R-module. Then M has no R-submodules other than (0) and M. So M is generated (as an R-module) by any nonzero element of M.

Fix any $em_0 \in eM$ for some $m_0 \in M$, such that $em_0 \neq 0$. Pick an arbitrary $em \in eM$. (From the problem statement, we know that every element of eM can be written in this way.)

Since $em_0 \neq 0$ and $em_0 \in eM \subseteq M$, we have that em_0 generates all of M. So in particular, $m = rem_0$ for some $r \in R$. Now $em = erem_0 = ere^2m_0 = (ere)em_0$. So eM is generated (as an eRe-module) by em_0. Since em_0 was chosen as an arbitrary nonzero element of eM, this means eM must be a simple eRe-module.

(iii) By Theorem 2.5, R is a semisimple ring if and only if ${}_RR$ is a semisimple R-module. Similarly, eRe is a semisimple ring if and only if ${}_{eRe}eRe$ is a semisimple eRe-module. Therefore it suffices to show that if ${}_RR$ is a semisimple R-module, then ${}_{eRe}eRe$ is a semisimple eRe-module.

Since our functor F is an additive functor and takes simple R-modules to either simple eRe-modules or to 0, we have that F must take semisimple R-modules to semisimple eRe-modules. Now Re is an R-module such that $F(Re) = eRe$. Moreover, Re is semisimple as an R-module since ${}_RR$ is semisimple. Therefore eRe is a semisimple eRe-module. So eRe is a semisimple ring.