MATH 6250: Theory of Rings
Homework 1
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4. (The matrix power functor.) Let R be a ring, $\varphi : M \to N$ be a homomorphism of R-modules, and n be a positive integer.

(i) Show that M^n is an $M_n(R)$-module, where the action of the ring on the module is that of multiplication of an $n \times n$ matrix by a column of length n.

(ii) Show that the map
$$\varphi^n : M^n \to N^n : \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} \mapsto \begin{bmatrix} \varphi(m_1) \\ \vdots \\ \varphi(m_n) \end{bmatrix}$$
of φ acting coordinatewise is an $M_n(R)$-module homomorphism.

(iii) Show that $M \mapsto M^n$, $\varphi \mapsto \varphi^n$ is a functor from ${}_R Mod$ to ${}_{M_n(R)}Mod$.

Solution.

(i) $M_n(R)$ is a ring with identity with matrix multiplication and addition, and M^n is an abelian group with vector addition. Let $r, s \in M_n(R)$ and $x, y \in M^n$. Then $r(x+y) = rx+ry$ and $(r+s)x = rx+sx$, because matrix multiplication and addition is distributive (considering the x, y vectors to be matrices). Also, $(rs)x = r(sx)$, because matrix multiplication is associative, furthermore $Ix = x$, where $I \in M_n(R)$ is the identity matrix. Thus M^n is an $M_n(R)$-module, as all the module axioms are true.

(ii) Let
$$x = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in M^n \text{ and } y = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in M^n.$$

Then
$$\varphi^n(x+y) = \varphi^n\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}\right) = \varphi^n\left(\begin{bmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{bmatrix}\right) = \begin{bmatrix} \varphi(a_1+b_1) \\ \vdots \\ \varphi(a_n+b_n) \end{bmatrix} =$$
$$= \begin{bmatrix} \varphi(a_1)+\varphi(b_1) \\ \vdots \\ \varphi(a_n)+\varphi(b_n) \end{bmatrix} = \begin{bmatrix} \varphi(a_1) \\ \vdots \\ \varphi(a_n) \end{bmatrix} + \begin{bmatrix} \varphi(b_1) \\ \vdots \\ \varphi(b_n) \end{bmatrix} =$$
$$= \varphi^n\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) + \varphi^n\left(\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}\right) = \varphi^n(x) + \varphi^n(y).$$

Now let

$$r = \begin{bmatrix} c_{11} & \dots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \dots & c_{nn} \end{bmatrix} \in M_n(R).$$

Then

$$\begin{aligned} \varphi^n(rx) &= \varphi^n \left(\begin{bmatrix} c_{11} & \dots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \dots & c_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right) = \varphi^n \left(\begin{bmatrix} c_{11}a_1 + \ldots + c_{1n}a_n \\ \vdots \\ c_{n1}a_1 + \ldots + c_{nn}a_n \end{bmatrix} \right) = \\ &= \begin{bmatrix} \varphi(c_{11}a_1 + \ldots + c_{1n}a_n) \\ \vdots \\ \varphi(c_{n1}a_1 + \ldots + c_{nn}a_n) \end{bmatrix} = \begin{bmatrix} c_{11}\varphi(a_1) + \ldots + c_{1n}\varphi(a_n) \\ \vdots \\ c_{n1}\varphi(a_1) + \ldots + c_{nn}\varphi(a_n) \end{bmatrix} = \\ &= \begin{bmatrix} c_{11} & \dots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \dots & c_{nn} \end{bmatrix} \begin{bmatrix} \varphi(a_1) \\ \vdots \\ \varphi(a_n) \end{bmatrix} = \begin{bmatrix} c_{11} & \dots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \dots & c_{nn} \end{bmatrix} \varphi^n \left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right) = r\varphi^n(x). \end{aligned}$$

This proves that φ^n is an $M_n(R)$-module homomorphism.

(iii) To prove that $M \mapsto M^n$, $\varphi \mapsto \varphi^n$ is a functor from ${}_R Mod$ to ${}_{M_n(R)}Mod$, there are four conditions that we have to check.

(1) Let $\varphi : M_1 \to M_2$, $\psi : M_2 \to M_3$ be homomorphisms of R-modules. We have to check that $(\psi \circ \varphi)^n = \psi^n \circ \varphi^n$. Again let

$$x = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in M_1^n.$$

Then

$$\begin{aligned} (\psi \circ \varphi)^n(x) &= (\psi \circ \varphi)^n \left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right) = \begin{bmatrix} (\psi \circ \varphi)(a_1) \\ \vdots \\ (\psi \circ \varphi)(a_n) \end{bmatrix} = \begin{bmatrix} \psi(\varphi(a_1)) \\ \vdots \\ \psi(\varphi(a_n)) \end{bmatrix} = \\ &= \psi^n \left(\begin{bmatrix} \varphi(a_1) \\ \vdots \\ \varphi(a_n) \end{bmatrix} \right) = \psi^n \left(\varphi^n \left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right) \right) = (\psi^n \circ \varphi^n)(x) \end{aligned}$$

Thus $(\psi \circ \varphi)^n = \psi^n \circ \varphi^n$.

(2) We have to check that $\mathrm{id}_M^n = \mathrm{id}_{M^n}$:

$$\mathrm{id}_M^n(x) = \mathrm{id}_M^n \left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right) = \begin{bmatrix} \mathrm{id}_M(a_1) \\ \vdots \\ \mathrm{id}_M(a_n) \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = x = \mathrm{id}_{M^n}(x)$$

(3) If F is the functor and $\varphi : M \to N$ is a homomorphism, then $F(\mathrm{dom}(\varphi)) = F(M) = M^n = \mathrm{dom}(\varphi^n) = \mathrm{dom}(F(\varphi))$, and

(4) $F(\mathrm{cod}(\varphi)) = F(N) = N^n = \mathrm{cod}(\varphi^n) = \mathrm{cod}(F(\varphi))$.

Thus $M \mapsto M^n$, $\varphi \mapsto \varphi^n$ is a functor.