rings1p3

C. Bridges, T. Gern

(One-sided ideals of $\mathrm{End}_{\mathbb{D}}(V)$.) Let M be an R-module, and let $S \subseteq M$ be a subset. The *annihilator* of S is $annS = \{r \in R \mid rS = \{0\}\}$.
Let V be a finite dimensional (left) $\mathbb{D}$-vector space. The purpose of this exercise is to determine the left and right ideals of $R = \mathrm{End}_{\mathbb{D}}(V)$.

(i) Show that $L = ann(U)$ is a left idea lof R for any subspace $U < V$.

Proof. We need to show that L is closed under elements of R acting on L on the left, and that the group structure of R, $(R, +)$, contains L as a subgroup. Suppose $r \in L$, let $-r$ be the additive inverse of r in R. Then $-rU = -1 \cdot rU = -1 \cdot \{0\} = \{0\}$, hence $-r \in L$. Suppose $r_1, r_2 \in L$. Then $(r_1 - r_2)U = \{(r_1 - r_2)u \mid u \in U\} = \{r_1 u - r_2 u \mid u \in U\} = \{0\}$. Now, for all $a \in R$ and $r \in L$, $(ar)U = a(rU) = a \cdot \{0\} = \{0\}$. Hence L is a left ideal for any subspace $U < V$. □

(ii) Show that if L is a left ideal of R, then $L = ann(U)$ for some subspace $U < V$.

Proof. First we will show that if $e, f \in R$ and $\ker(f) \not\subseteq \ker(e)$, then there is a $d \in R$ such that $\ker(de + f)$ is properly contained in $\ker(f)$. Let $x \in \ker(f) \setminus \ker(e)$ and let $w \in V \setminus \mathrm{im}(f)$. Such a w must exist otherwise f is surjective, so by the Rank Nullity Theorem, $\ker(f) = 0$. This means $\ker(f) \subseteq \ker(e)$, a contradiction. Now, extend $v_1 = e(x)$ to a basis $v_1, \ldots, v_n$ of V. Define an endomorphsim $d : V \to V$ by

$$d(v_i) = \begin{cases} w & \text{if } i = 1; \\ 0 & \text{else.} \end{cases}$$

Then $(de + f)(x) = w \neq 0$, but $f(x) = 0$. Furthermore, if $(de + f)(y) = 0$ then for some $b \in \mathbb{D}$ we have

$$\begin{aligned} -de(y) &= f(y) \\ b \cdot w &= f(y) \end{aligned}$$

which means $b = 0$ since $w \notin \mathrm{im}(f)$. Then $y \in \ker(f)$, so $\ker(de + f) \subsetneq \ker(f)$.

Now we will show that $L = ann(U)$ for some subspace $U < V$. Let L be a left ideal. Since the dimension of the kernel of any element of L is an integer between 0 and $\dim(V)$, there exists a minimal dimension of kernels of elements of L. Let f be an element of L whose kernel has this dimension. Suppose there is some e in L such that $\ker(f) \not\subseteq \ker(e)$. Then by the above we see that there is some d in R such that $\ker(de + f) \subsetneq \ker(f)$. Since L is an ideal and f and e are in L, we see that $de + f \in L$ which contradicts the minimality of the dimension of $\ker(f)$. Therefore for all $e \in L$, $\ker(f) \subseteq \ker(e)$. It follows that $L \subseteq ann(\ker(f))$. Clearly $R \cdot f \subseteq L$, so we have that

$$R \cdot f \subseteq L \subseteq ann(\ker(f)).$$

If we show that $ann(\ker(f)) \subseteq R \cdot f$ then we are done.
To see that this is true, first note that by the Rank-Nullity Theorem, for all $h, f \in L$,

$$\dim(\ker(f)) + \dim(\mathrm{Im}(f)) = \dim(V) = \dim(\ker(h)) + \dim(\mathrm{Im}(h)),$$

so $\dim(\mathrm{Im}(h)) \leq \dim(\mathrm{Im}(f))$ if $h \in ann(\ker(f))$.
Let $h \in ann(\ker(f))$, $\dim(\mathrm{Im}(h)) = s \leq r = \dim(\mathrm{Im}(f))$. Now let $\{x_{r+1}, \dots, x_n\}$ be a basis for $\ker(f)$. Extend this to a basis $\{x_{s+1}, \dots, x_n\}$ for $\ker(h)$. Then extend this to a basis $\{x_1, \dots, x_n\}$ for V. By writing each element of V as a linear sum of basis vectors, we see that the set of non-zero $f(x_i)$ span the image of f. Also, we see that $\mathrm{span}\{x_i \mid f(x_i) = 0\}$ is contained in $\ker(f)$. By the Rank-Nullity Theorem we see that the set of non-zero $f(x_i)$ form a basis for $\mathrm{Im}(f)$ and $\mathrm{span}\{x_i \mid f(x_i) = 0\} = \ker(f)$. Similarly, the set of non-zero $h(x_i)$ form a basis for $\mathrm{Im}(h)$ and $\mathrm{span}\{x_i \mid h(x_i) = 0\} = \ker(h)$.
Now we have $\{f(x_i) \mid f(x_i) \neq 0\} = \{f(x_1), \dots, f(x_r)\}$ and $\{h(x_i) \mid h(x_i) \neq 0\} = \{h(x_1), \dots, h(x_s)\}$. Let $f(x_i) = v_i$ for $i = 1 \dots r$ and extend this to a basis $\{v_1, \dots, v_n\}$ of V. Similarly, let $h(x_i) = w_i$ for $i = 1 \dots s$ and extend this to a basis $\{w_1, \dots, w_n\}$ of V.
Now let $r : V \to V$ be an endomorphsim defined as follows:

$$r(v_i) = \begin{cases} w_i & \text{if } i = 1 \dots s; \\ 0 & \text{else,} \end{cases}$$

Since the v_i form a basis for V, we see that r is well-defined and

$$r(f(x_i)) = r(v_i) = \begin{cases} w_i & \text{if } i = 1 \dots s \\ 0 & \text{else} \end{cases} = \begin{cases} h(x_i) & \text{if } i = 1 \dots s \\ 0 & \text{else} \end{cases} = h(x_i).$$

Therefore, for all $h \in ann(\ker(f))$, there exists an $r \in R$ such that $h = rf$ for all $v \in V$. Hence $ann(\ker(f)) \subseteq R \cdot f$. $\square$

(iii) Determine a similar correspondence between the right ideals of R and subspaces of V.

Let U be a subspace of V. Define the set of all endomorphsims whose image is in U to be $\mathrm{Im}(U) := \{r \in R \mid r(V) \subseteq U\}$. $\mathrm{Im}(U)$ is an additive subgroup since U is a subspace: For all $r_1, r_2 \in \mathrm{Im}(U)$

$$(r_1 - r_2)(V) = \{r_1(v_1) - r_2(v_2) \mid v_1, v_2 \in V\} \subseteq U,$$

so $r_1 - r_2 \in \mathrm{Im}(U)$. Also, $\mathrm{Im}(U)$ is closed under multiplication on the right by elements of R since for all $r, s \in R$, $r(V) \subseteq r(s(V))$. Therefore $\mathrm{Im}(U)$ is a right ideal of R.

Claim: If I is a right ideal of R, then $I = \mathrm{Im}(U)$ for some $U \subset V$.

Proof. First we will show that for any $e, f \in R$ with $\mathrm{Im}(e) \not\subseteq \mathrm{Im}(f)$, there is a $d \in R$ such that $\mathrm{Im}(f) \subsetneq \mathrm{Im}(ed + f)$. Let $v \in \mathrm{Im}(e) \setminus \mathrm{Im}(f)$. So for some $w \in V$, $e(w) = v$ but clearly $v \notin \mathrm{Im}(f)$. Let $\{v_1, \dots, v_l\}$ be a basis for $\ker(f)$, extend this to a basis $\{v_1, \dots, v_n\}$ of V. Define an endomorphism $d : V \to V$ by

$$d(v_i) = \begin{cases} w & \text{if } i = 1, \dots, l; \\ 0 & \text{else.} \end{cases}$$

Now

$$(ed+f)(v_i) = ed(v_i)+f(v_i) = \begin{cases} ed(v_i) & \text{if } i = 1,\dots,l, \\ f(v_i) & \text{else.} \end{cases} = \begin{cases} v & \text{if } i = 1,\dots,l, \\ f(v_i) & \text{else.} \end{cases}$$

Suppose $a \in \mathrm{Im}(f)$. Then there is some $b \in V$ such that $f(b) = a$. Writing b as a linear combination of basis vectors $v_1, \dots, v_n$, we have

$$\begin{aligned} a &= f(b) \\ &= f(\sum_{i=1}^{n} \alpha_i v_i) \\ &= f(\sum_{i=l+1}^{n} \alpha_i v_i) \\ &= (ed+f)(\sum_{i=l+1}^{n} \alpha_i v_i), \end{aligned}$$

and therefore $\mathrm{Im}(f) \subseteq \mathrm{Im}(ed+f)$. It is clear from our definition of d that $v \in \mathrm{Im}(ed+f) \setminus \mathrm{Im}(f)$, so $\mathrm{Im}(f) \subsetneq \mathrm{Im}(ed+f)$.

Since V is finite dimensional, there is a maximum dimension of images of elements of I. Select an $f \in I$ such that the dimension of the image, U, of f is maximal. Suppose there is some $e \in I$ such that $\mathrm{Im}(e) \not\subseteq \mathrm{Im}(f)$. We have seen that this contradicts the maximality of the dimension of f since $ed+f \in I$ because I is a right ideal. Hence $\mathrm{Im}(e) \subseteq \mathrm{Im}(f)$ for all $e \in I$. Therefore $I \subseteq \mathrm{Im}(U)$.

Now let $e \in \mathrm{Im}(U)$. Let $\{w_{j+1}, \dots, w_n\}$ be a basis for $\ker(e)$. Extend this to a basis $\{w_1, \dots, w_n\}$ of V. For $1 \le i \le j$, $e(w_i)$ are linearly independent and non-zero. By checking dimensions, we see that they form a basis of $\mathrm{Im}(f)$. Since $\mathrm{Im}(e) \subseteq U = \mathrm{Im}(f)$, there are $v_1, \dots, v_j \in V$ such that $f(v_i) = e(w_i)$ for $i = 1, \dots, j$. Since these $e(w_i)$ are linearly independent, so are the v_i. Extend the v_i to a basis of V, $\{v_1, \dots, v_n\}$. Define an endomorphism $r : V \to V$ as

$$r(w_i) = \begin{cases} v_i & \text{if } i = 1,\dots,j; \\ 0 & \text{else.} \end{cases}$$

Now r is defined so that $e(w_i) = fr(w_i)$ for all $i = 1, \dots, n$. Since e was arbitrary, we have $\mathrm{Im}(U) \subseteq f \cdot R$. In total we have

$$f \cdot R \subseteq I \subseteq \mathrm{Im}(U) \subseteq f \cdot R$$

Therefore $I = \mathrm{Im}(U)$. □

(iv) Can anything interesting be said about the case where V is infinite dimensional?

Let $V = \mathrm{Span}\{v_i \mid i \in \mathbb{N}\}$. Then the proof in part (i) still applies to V, so annihilators of subspaces of V are left ideals. Also, $\mathrm{Im}(U)$ will be a right ideal for any subspace U of V.
An immediate obstacle in proving the converse in the infinite dimensional case is

that there may be no minimal dimension of kernels of elements of a left ideal, or maximal dimension of images of elements of a right ideal. For example, let

$$I_i = \{e \in R \mid e(v_j) = 0 \text{ for all } j > i\},$$

and

$$I = \bigcup_{i \in \mathbb{N}} I_i.$$

Then if $r_1, r_2 \in I$, then for all $a \in R$, $ar_1 \in I$ and $r_1 - r_2 \in I$, so I is a left ideal. But for any non zero subspace $U \subseteq V$, U contains a finite linear combination of the v_i, so not every element of I annihilates all of U. If $U = \{0\}$, then since $I \neq R$, I is not the annihilator of U.

If we speak only of left ideals with the property that they do have a minimal dimension of kernels of its elements, or of right ideals that do have a maximal dimension of images of its elements, then there also needs to be an adjustment to the proof of the existence of the d found in the above proofs. For example, the proof of the existence of d in (ii) was based on finding an element w not in the image of f. This w may not exist if V is infinite dimensional: Let $V = \text{Span}\{v_i \mid i \in \mathbb{N}\}$. Define f by

$$f(v_i) = \begin{cases} 0 & \text{if } i = 1; \\ v_{i-1} & \text{else.} \end{cases}$$

Then $\text{im}(f) = V$ and $\ker(f) = \text{Span}(v_1)$. The constructive proof of d in (ii) does not translate to this f, a new method for finding this d would need to be found, or possibly a new condition on which element would be have a kernel properly contained in the kernel of f. For example, maybe given e and f such that $\ker(f) \not\subseteq \ker(e)$ there are a d and c such that $\ker(de + cf) \subsetneq \ker(f)$.