Problem 2 Scott Andrews, Clifford Blakestad

(i) Let $R = M_n(T)$ for some ring T, and let $E_{ij} \in R$ be the matrix whose ij-entry is one and whose other entries are 0. Show that $\{E_{ij} \mid 1 \leq i, j \leq n\}$ is a set of matrix units for R.

If $A, B \in R$, note that

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

Let $I = \sum_{i=1}^{n} E_{ii}$. Then for $A \in R$,

$$(AI)_{ij} = \sum_{k=1}^{n} A_{ik} I_{kj} = A_{ij} I_{jj} = A_{ij}.$$

Similarly, $(IA)_{ij} = A_{ij}$, and I is the multiplicative identity of R.

Now consider

$$\begin{aligned}(E_{ij}E_{kl})_{ab} &= \sum_{c=1}^{n} (E_{ij})_{ac} (E_{kl})_{cb} \\ &= (E_{ij})_{aj} (E_{kl})_{jb} \\ &= \delta_{ai} \delta_{jk} \delta_{bl}.\end{aligned}$$

The matrix A with $A_{ab} = \delta_{ai}\delta_{bl}$ is E_{il}, hence $E_{ij}E_{kl} = \delta_{jk} E_{il}$.

(ii) Show that if a ring S has a set of matrix units $\{e_{ij} \mid 1 \leq i, j \leq n\}$ then $S \cong M_n(S')$ for some subring S' of S.

Lemma. Let S be a ring with a set of matrix units $\{e_{ij} \mid 1 \leq i, j \leq n\}$, and let R be a subring of S such that each $r \in R$ commutes with each matrix unit of S. Then the map

$$\begin{aligned} f : M_n(R) &\to S \\ (r_{ij}) &\mapsto \sum_{1 \leq i,j \leq n} r_{ij} e_{ij} \end{aligned}$$

is an embedding.

Proof. It is clear that f is a homomorphism of abelian groups and maps the identity to the identity. To check that f respects multiplication, note that

$$\begin{aligned} f((r_{ij})(s_{kl})) &= f\left(\left(\left(\sum_{1 \leq k \leq n} r_{ik} s_{kj} \right)_{ij} \right) \right) \\ &= \sum_{1 \leq i,j \leq n} \left(\sum_{1 \leq k \leq n} r_{ik} s_{kj} \right) e_{ij} \end{aligned}$$

and

$$\begin{aligned}
f((r_{ij}))f((s_{kl})) &= \sum_{1\le i,j\le n} r_{ij}e_{ij} \sum_{1\le k,l\le n} s_{kl}e_{kl} \\
&= \sum_{1\le i,j,k,l\le n} r_{ij}e_{ij}s_{kl}e_{kl} \\
&= \sum_{1\le i,j,k,l\le n} r_{ij}s_{kl}e_{ij}e_{kl} \\
&= \sum_{1\le i,j,l\le n} r_{ij}s_{jl}e_{il} \\
&= \sum_{1\le i,l\le n} \left(\sum_{1\le j\le n} r_{ij}s_{jl} \right) e_{il}.
\end{aligned}$$

Finally, assume that $f((r_{ij})) = 0$; then

$$\sum_{1\le i,j\le n} r_{ij}e_{ij} = 0.$$

Fix a and b, and consider

$$\begin{aligned}
0 &= \sum_{1\le k\le n} e_{ka} \left(\sum_{1\le i,j\le n} r_{ij}e_{ij} \right) e_{bk} \\
&= \sum_{1\le k,j\le n} r_{aj}e_{kj}e_{bk} \\
&= \sum_{1\le k\le n} r_{ab}e_{kk} \\
&= r_{ab}.
\end{aligned}$$

It follows that $(r_{ij}) = 0$, and f is injective.

$\square$

Now define $S' = C_S(\{e_{ij} \mid 1 \le i, j \le n\})$; that is, S' is the subring of S consisting of all elements which commute with all of the matrix units. By the lemma, the map

$$\begin{aligned}
f : M_n(S') &\to S \\
(s_{ij}) &\mapsto \sum_{1\le i,j\le n} s_{ij}e_{ij}
\end{aligned}$$

is an embedding. It suffices to show that f is surjective.

For $s \in S$, define

$$s_{ij} = \sum_{k=1}^{n} e_{ki}se_{jk}.$$

Note that

$$\begin{aligned} s_{ij}e_{ab} &= e_{ai}se_{jb} \\ &= e_{ab}s_{ij} \end{aligned}$$

hence $s_{ij} \in S'$. Furthermore,

$$\begin{aligned} f((s_{ij})) &= \sum_{1\le i,j\le n} s_{ij}e_{ij} \\ &= \sum_{1\le i,j\le n} e_{ii}se_{jj} \\ &= \sum_{i=1}^{n} e_{ii}\left(s\sum_{j=1}^{n} e_{jj}\right) \\ &= \sum_{i=1}^{n} e_{ii}s \\ &= s \end{aligned}$$

hence f is surjective. It follows that f is an isomorphism, and $S \cong M_n(S')$.

(iii) Show that a homomorphic image of $M_n(S)$ is isomorphic to $M_n(S/I)$ for some ideal I of S.

Let $f : M_n(S) \to R$ be a surjective homomorphism. Let i be the inclusion

$$\begin{aligned} i : S &\hookrightarrow M_n(S) \\ s &\mapsto sI \end{aligned}$$

and let R' be the image of $f \circ i$. As $R' \cong S/\ker(f \circ i)$, it suffices to show that $M_n(R') \cong R$.

Note that $\{f(E_{ij}) \mid 1 \le i, j \le n\}$ is a set of matrix units of R. For $r \in R'$, if $r = (f \circ i)(s)$, then

$$\begin{aligned} rf(E_{ij}) &= f((sI)E_{ij}) \\ &= f(E_{ij}(sI)) \\ &= f(E_{ij})r. \end{aligned}$$

It follows that all elements of R' commute with the matrix units of R, and by the lemma the homomorphism

$$\begin{aligned} g : M_n(R') &\to R \\ (r_{ij}) &\mapsto r_{ij}f(E_{ij}) \end{aligned}$$

is injective. Furthermore, for $r \in R$ with $r = f((s_{ij}))$, $(s_{ij}) \in M_n(S)$,

$$\begin{aligned} r &= f((s_{ij})) \\ &= f\left(\sum_{1\le i,j\le n} (s_{ij}I)E_{ij}\right) \end{aligned}$$

hence g is surjective, and $M_n(R') \cong R$.