Problem 1 Scott Andrews, Clifford Blakestad

Prove a Cayley-type theorem which faithfully represents an arbitrary small category as a subcategory of the category of sets.

Proposition Let $\mathcal{C}$ be a small category. Define

$$\begin{aligned} F : \mathcal{C} &\to \mathcal{SET} \\ A &\mapsto \{f \in \mathrm{Mor}(\mathcal{C}) \mid \mathrm{cod}(f) = A\} \end{aligned}$$

for $A \in \mathrm{Ob}(\mathcal{C})$, and for $g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $f \in F(A)$,

$$F(g)(f) = g \circ f.$$

Then F is a covariant functor which is injective as a function on $\mathrm{Ob}(\mathcal{C})$ and on $\mathrm{Mor}(\mathcal{C})$.

Proof. Let $f \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$; then for $h \in F(A)$,

$$\begin{aligned} F(f \circ g)(h) &= (f \circ g) \circ h \\ &= f \circ (g \circ h) \\ &= f \circ F(g)(h) \\ &= F(f)(F(g)(h)) \\ &= (F(f) \circ F(g))(h) \end{aligned}$$

hence $F(f \circ g) = F(f) \circ F(g)$. If $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, then

$$F(f) \circ F(\mathrm{id}_A) = F(f \circ \mathrm{id}_A) = F(f)$$

and

$$F(\mathrm{id}_B) \circ F(f) = F(\mathrm{id}_B \circ f) = F(f)$$

so F preserves the identity maps. Finally, it is clear by the definition of F that

$$F(\mathrm{dom}(f)) = \mathrm{dom}(F(f))$$

and

$$F(\mathrm{cod}(f)) = \mathrm{cod}(F(f))$$

hence F is a covariant functor.

As $\mathrm{cod} : \mathrm{Mor}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$ is a function, it is clear that F is injective as a function on $\mathrm{Ob}(\mathcal{C})$. Let $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$; if $F(f) = F(g)$, then

$$\begin{aligned} F(f)(\mathrm{id}_A) &= F(g)(\mathrm{id}_A) \\ f \circ \mathrm{id}_A &= g \circ \mathrm{id}_A \\ f &= g \end{aligned}$$

and F is injective as a function on $\mathrm{Mor}(\mathcal{C})$.

$\square$