Problem. Let $B_1, \ldots, B_n$ be left ideals (resp. ideals) in a ring R. Show that $R = B_1 \oplus \cdots \oplus B_n$ iff there exist idempotents (resp. central idempotents) $e_1, \ldots, e_n$ with sum 1 such that $e_i e_j = 0$ whenever $i \neq j$, and $B_i = Re_i$ for all i.

In the case where the B_i's are ideals, if $R = B_1 \oplus \cdots \oplus B_n$, then each B_i is a ring with identity e_i, and we have an isomorphism between R and the direct product of rings $B_1 \times \cdots \times B_n$. Show that any isomorphism of R with a finite direct product of rings arises in this way.

Solution. ($\Rightarrow$) Assume that $R = B_1 \oplus \cdots \oplus B_n$ for some left ideals B_i. Therefore each $r \in R$ can be written uniquely as $b_1 + b_2 + \ldots + b_n$, with $b_i \in B_i$ for each i. In particular, there is a unique way of writing 1 as the sum of elements of these B_i. Let

$$1 = e_1 + \cdots + e_n,$$

where $e_i \in B_i$.

We know that $Re_i \subseteq B_i$ because $e_i \in B_i$ and the B_i are left ideals.

Let $b \in B_i$. Then $b = b \cdot 1 = be_1 + \cdots + be_n$. Since each $be_j \in B_j$, and the way of writing b as a sum of elements from the various B_j is unique, it must be that $be_i = b$ and $be_j = 0$ for $j \neq i$. The first consequence is that $b = be_i \in Re_i$, and therefore $B_i \subseteq Re_i$. Since we already demonstrated the other containment, now we have $B_i = Re_i$.

Replacing b with e_i in the computation above, we see that $e_i^2 = e_i$ and that $e_i e_j = 0$ for $i \neq j$.

In the special case where the B_i are two-sided ideals, we have additionally that $e_i R \subseteq B_i$, and if $b \in B_i$ then $b = e_1 b + e_2 b + \ldots + e_n b$, giving (by uniqueness of writing b) that $e_i b = b$ and $e_j b = 0$ for $j \neq i$. To see that the e_i are central, let $r \in R$, and write $r = b_1 + b_2 + \ldots + b_n$ where $b_i \in B_i$ for each i. Then

$$e_i r = e_i b_1 + e_i b_2 + \ldots + e_i b_n = e_i b_i = b_i = b_i e_i = b_1 e_i + b_2 e_i + \ldots + b_n e_i = re_i.$$

($\Leftarrow$) Now assume that R has a set of left ideals $B_1, \ldots, B_n$, along with elements $e_i \in B_i$ such that $B_i = Re_i$, the sum $e_1 + \ldots + e_n = 1$ and $e_i e_j = 0$ for $i \neq j$.

The sum $B_1 + \ldots + B_n$ is a left ideal containing $e_1 + \ldots + e_n = 1$, and hence must equal R.

Let $r \in R$ and suppose that $r = b_1 + b_2 + \ldots + b_n = c_1 + c_2 + \ldots + c_n$, where $b_i, c_i \in B_i$. Then $0 = (b_1 - c_1) + (b_2 - c_2) + \ldots + (b_n - c_n)$. Note $b_i - c_i \in B_i$ for each i. Since $B_i = Re_i$ by assumption, we can write $b_i - c_i = r_i e_i$ for some $r_i \in R$. So $0 = r_1 e_1 + r_2 e_2 + \ldots + r_n e_n$. Therefore $0 = (r_1 e_1 + \cdots + r_n e_n) e_i = 0$ for any fixed i. But this simplifies out to $0 = r_i e_i^2 = r_i e_i = b_i - c_i$. So $b_i = c_i$.

Therefore r is written uniquely as a sum of elements of the B_i, and $R = B_1 \oplus \cdots \oplus B_n$.

In the special case where the e_i are central idempotents, we have $B_i = Re_i = e_i R$ are two-sided ideals. $\square$

In the case where the B_i's are ideals, if $R = B_1 \oplus \cdots \oplus B_n$, then each B_i is a ring with identity e_i, and we have an isomorphism between R and the direct product of rings $B_1 \times \cdots \times B_n$.

Now suppose that $R = B_1 \oplus \cdots \oplus B_n$ where the B_i are ideals. We already showed that if $b \in B_i$ then $b = be_i = e_ib$ (since the e_i are central in this case). So e_i acts as a two-sided multiplicative identity on the ideal B_i.

Define $\varphi : R \to B_1 \times \cdots \times B_n$ by $\varphi(r_1 + \cdots + r_n) = (r_1, \ldots, r_n)$, where $r_i \in B_i$. Clearly φ is a bijection since any element of R can be written uniquely as $r_1 + \cdots + r_n$, with $r_i \in B_i$.

Claim. φ is a ring isomorphism.

If $r, s \in R$ with $r = r_1 + \cdots + r_n$ and $s = s_1 + \cdots + s_n$, with $r_i, s_i \in B_i$, then

$$\begin{aligned}
\varphi(r+s) &= \varphi(r_1 + \cdots + r_n + s_1 + \cdots + s_n) \\
&= \varphi(r_1 + s_1 + r_2 + \cdots + r_n + s_n) \\
&= (r_1 + s_1, \ldots, r_n + s_n) \\
&= (r_1, \ldots, r_n) + (s_1, \ldots, s_n) \\
&= \varphi(r) + \varphi(s).
\end{aligned}$$

and

$$\begin{aligned}
\varphi(rs) &= \varphi((r_1 + \cdots + r_n)(s_1 + \cdots + s_n)) \\
&= \varphi\left(\sum_{i=1}^{n} r_i(s_1 + \ldots + s_j)\right) \\
&= \varphi\left(\sum_{i=1}^{n} r_i s_i\right) && \text{since } r_i s_j = 0 \text{ when } i \neq j \\
&= \varphi\left(r_1 s_1 + \cdots + r_n s_n\right) && \text{where } r_i s_i \in B_i \text{ for each } i \\
&= (r_1 s_1, \ldots, r_n s_n) \\
&= (r_1, \ldots, r_n)(s_1, \ldots, s_n) \\
&= \varphi(r)\varphi(s).
\end{aligned}$$

Moreover, the multiplicative identity in $B_1 \oplus \cdots \oplus B_n$ is $(e_1, \ldots, e_n)$, and one of our assumptions gives that

$$\varphi(1_R) = \varphi(e_1 + \cdots + e_n) = (e_1, \ldots, e_n).$$

So φ is a ring homomorphism, and has inverse given by $(r_1, \cdots, r_n) \mapsto r_1 + \cdots + r_n$. So R is isomorphic to $B_1 \times \cdots \times B_n$. $\square$

Any isomorphism of R with a finite direct product of rings arises in this way.

Let $R, A_1, \ldots, A_n$ be rings and suppose φ is a ring isomorphism from $A_1 \times \cdots \times A_n$ to R.

For each i, define $a_i \in A_1 \times \cdots \times A_n$ by $a_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 occurs in the i^{th} place. Note that for each i, we have

- $a_i^2 = a_i$,
- $a_i a_j = 0$ when $i \neq j$,
- a_i is central in $A_1 \times \cdots \times A_n$, and
- $\sum_{i=1}^{n} a_i = (1, \ldots, 1)$ acts as a multiplicative identity in $A_1 \times \cdots \times A_n$.

For each i, define $e_i \in R$ by $e_i = \varphi(a_i)$. Since φ is an isomorphism, for each i we immediately have

- $e_i^2 = e_i$,
- $e_i e_j = 0$ when $i \neq j$,
- e_i is central in R, and
- $\sum_{i=1}^{n} e_i$ acts as a multiplicative identity in R.

Define $B_i = Re_i$ for each i. Since the e_i are central in R, these B_i are two-sided ideals of R as desired. The first result now tells us that $R = B_1 \oplus \cdots \oplus B_n$.