Problem. *Let R be a domain. If R has a minimal left ideal, show that R is a division ring. (In particular, a left Artinian domain must be a division ring.)*

Solution. Let R be a domain and let I be a minimal non-zero left ideal of R. Pick $b \in I$ such that $Ib \neq 0$. Then $Rb \subseteq I$ is a left ideal, so $Rb = I$. We also have $b^2 \in I$, so by a similar argument $I = Rb^2$. Now we have $Rb = Rb^2$, so $b \in Rb^2$, thus we can find $a \in R$ such that $b = ab^2$. Then since R is a domain we can cancel to obtain $ab = 1$. Then $ab = 1 \in I$, so $I = R$, thus the only left ideals of R are R and $\{0\}$, so R is a division ring.

$\square$