Theory of Rings Homework

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Rings1p1.16: Let K be a division ring with center k.

(1) Show that the center of the polynomial ring $R = K[x]$ is $k[x]$.

Proof: We'll show that the two are subsets of each other. Let $c(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 \in k[x]$. Let $a(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 \in K[x]$. Then, $a(x)c(x) = a_m c_n x^{m+n} + (a_m c_{n-1} + a_{m-1}c_n)x^{m+n-1} + \cdots + (a_{m+n-i}c_0 + a_{m+n-i-1}c_1 + \cdots + a_1 c_{m+n-i-1} + a_0 c_{m+n-i})x^{m+n-i} + \cdots + (a_1 c_0 + a_0 c_1)x + a_0 c_0 = c_n a_m x^{nm} + (c_n a_{m-1} + c_{n-1}a_m)x^{n+m-1} + \cdots + (c_{n+m-i}a_0 + c_{n+m-i-1}a_1 + \cdots + c_1 a_{n+m-i-1} + c_0 a_{m+n-i})x^{n+m-i} + \cdots + (c_1 a_0 + c_0 a_1)x + c_0 a_0 = c(x)a(x)$. Thus $k[x] \subseteq Z(R)$.

Now, assume that there is an element $p(x) = p_n x^n + \cdots + p_1 x + p_0 \in Z(R)$ with some $p_i \notin k$. Then there is an element $b \in K$ such that $p_i b \neq b p_i$. Then $bp(x) = bp_n x^n + \cdots + bp_i x^i + \cdots + bp_1 x + bp_0 \neq p_n b x^n + \cdots + p_i b x^i + \cdots + p_1 x + p_0 = p(x)b$ because of the ith term. Thus, $Z(R) \subseteq k[x]$, and therefore, the two are equal.

(2) For any $a \in K \setminus k$, show that the ideal generated by $x - a$ in $K[x]$ is the unit ideal.

Proof: Because $a \in K \setminus k$, there exists an element $b \in K$ such that $ab \neq ba$. The ideal generated by $x - a$ is closed under left and right multiplication from $K[x]$ and subtraction. Thus, $b(x-a)$ and $(x-a)b$ are in the ideal generated by $x - a$. And also, $b(x-a) - (x-a)b = bx - ba - bx + ab = ab - ba$ is in the ideal. Because a, b don't commute, $ab - ba \neq 0$. Because we are in a division ring, $ab - ba$ must have an inverse. Because $ab - ba$ is in the ideal, and the ideal is closed under multiplication, $(ab - ba)(ab - ba)^{-1} = 1$ is in the ideal. Thus, the ideal generated by $x - a$ is the same as the ideal generated by 1. Thus, this is a unit ideal.

(3) Show that any ideal $I \subseteq R$ has the form $R \cdot h$ where $h \in k[x]$.

Proof: Let I be an ideal of $K[x]$. Let $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a minimal degree polynomial in I. We can assume this polynomial is monic by multiplying through by the inverse of the leading coefficient. If $a_i \in k$ for all $i = 1, ..., n-1$, then we are done. If not, then there is some $a_j \notin k$. Let $b \in K$ such that $a_j b \neq b a_j$. Then $b(x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) = bx^n + ba_{n-1}x^{n-1} + \cdots + ba_j x^j + \cdots + ba_1 x + ba_0 \in I$ and $(x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)b = bx^n + a_{n-1}bx^{n-1} + \cdots + a_j b x^j + \cdots + a_1 bx + a_0 b \in I$, so their difference must be too. Thus, $(ba^{n-1} - ab^{n-1})x^{n-1} + \cdots + (ba_j - a_j b)x^j + \cdots + (ba_1 - a_1 b)x + ba_0 - a_0 b \in I$ We don't know the exact degree of this polynomial, but we do know that it is at least j because $ba_j \neq a_j b$ and at most $n - 1$. Thus we have found a polynomial of smaller degree in I, which is a contradiction. Thus, every a_i must be in k for $i = 1, \ldots, n-1$. Notice that polynomial rings over division rings are principal ideal domains. Indeed, let I be an ideal. Let $f(x)$ be a minimal degree monic polynomial in I. If $\deg(f) = 0$, then because K is a division ring, $(f) = I$. If $\deg(f) > 0$, then by the above argument, $f(x) \in k[x]$, the center of R. Hence, by the division algorithm, we see that $f(x)$ must divide any $g(x) \in I$ and because $f(x)$ is in the center, $f(x)$ will divide $g(x)$ from both the right and the left. Thus, we know that each ideal will be generated by one element. Therefore, every ideal is generated by some $h \in k[x]$.