Theory of Rings Homework

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Rings1p1.15(a)(b): Let $A = \mathbb{C}[x;\sigma]$, where σ denotes complex conjugation.

a) Show that $Z(A) = \mathbb{R}[x^2]$.

b) Show that $A/A\cdot(x^2+1)$ is isomorphic to $\mathbb{H}$, the division ring of the real quaterions.

Proof:

a) Claim: $\mathbb{R}[x^2] \subseteq Z(A)$. Let $a(x) = \sum_{j=0}^{n}(a_j + ib_j)x^j \in A$. Let $c(x) = \sum_{k=0}^{m} c_{2k}x^{2k} \in \mathbb{R}[x^2]$. Then $a(x)c(x) = \sum_{j+2k=0}^{n+2m}(a_j+ib_j)\sigma^j(c_{2k})x^{j+2k} = \sum_{j+2k=0}^{n+2m}(a_j+ib_j)c_{2k}x^{j+2k} = \sum_{j+2k=0}^{n+2m} c_{2k}(a_j+ib_j)x^{2k+j} = \sum_{j+2k=0}^{n+2m} c_{2k}\sigma^{2k}(a_j+ib_j)x^{2k+j} = (\sum_{2k=0}^{2m} c_{2k}x^{2k})(\sum_{j=0}^{m}(a_j+ib_j)x^j) = c(x)a(x)$.

$Z(A) \subseteq \mathbb{R}[x^2]$. Let $a(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_jx^j + \cdots + a_1x + a_0$ be a polynomial in A such that a_j has a non-zero complex part (i.e. $a_j = b + ic$ where $c \neq 0$). Then $xa(x) = (xa_nx^n + xa_{n-1}x^{n-1} + \cdots + xa_jx^j + \cdots + xa_1x + xa_0 = \sigma(a_n)x^{n+1} + \sigma(a_{n-1})x^n + \cdots + \sigma(a_j)x^{j+1} + \cdots + \sigma(a_1)x^2 + \sigma(a_0)x$. And $a(x)x = a_nx^{n+1} + a_{n-1}x^n + \cdots + a_jx^{j+1} + \cdots + a_1x^2 + a_0x$. If these two polynomials are equal, their x^{j+1}th coefficients must be equal . But if we compare the coefficient of x^{j+1}, we find the two terms to not be equal because the complex part of $a_j \neq 0$. Thus, any polynomial with at least one non-real coefficient will not be central. To show that any polynomial with an odd degreed term will not be in the center, let $b(x) = b_nx^n + \cdots + b_kx^k + \cdots + b_2x^2 + b_0$ be a polynomial where the kth term is odd, and let $a + ic \in \mathbb{C}$ such that $c \neq 0$. Then $b(x)(a+ic) = b_n\sigma^n(a+ic)x^n + \cdots + b_k\sigma^k(a+ic)x^k + \cdots + b_2\sigma^2(a+ic) + b_0(a+ic)$. And $(a+ic)b(x) = (a+ic)b_nx^n + \cdots + (a+ic)b_kx^k + \cdots + (a+ic)b_2x^2 + (a+ic)b_0$. Comparing the kth terms, we want $b_k\sigma^k(a+ic) = b_k(a+ic)$. Because this is in $\mathbb{C}$, and $b_k \neq 0$, we can cancel the b_k's. But because k is odd, and complex conjugation is self-inverting, we have $\sigma(a+ic) = a - ic \neq a + ic$. Thus, the kth terms are not equal, and so $b(x)$ is not in the center. Thus, we have $Z(A) \subseteq \mathbb{R}[x^2]$ and therefore, we have equality.

b) Any element in $A/A\cdot(x^2+1)$ can be represented uniquely by an element in A having the form $ax+b$, where $a, b \in \mathbb{C}$. This is because we can always use the division algorithm to divide any polynomial having degree 2 or higher by the polynomial x^2+1 to get a remainder of the form $ax+b$. This means that any element of $A/A\cdot(x^2+1)$ can be represented uniquely by an element of the form $p + qi + rx + six$, where $p, q, r, s \in \mathbb{R}$. Thus $A/A\cdot(x^2+1)$ is a four dimensional vector space over $\mathbb{R}$ having basis $\{1, i, x, ix\}$. On the other hand $\mathbb{H}$ is also a four dimensional vector space over $\mathbb{R}$ with basis $\{1, i, j, k\}$. Thus if we identify the elements $\{1, i, x, ix\}$ in $A/A\cdot(x^2+1)$ with the elements $\{1, i, j, k\}$ in $\mathbb{H}$, we get a vector space isomorphism and thus an additive group isomorphism between $A/A\cdot(x^2+1)$ and $\mathbb{H}$. To see that this correspondence is a ring isomorphism, we only need to check if it keeps the multiplicative structure. Multiplication in $\mathbb{H}$ is defined by the identities $i^2 = -1$, $j^2 = -1$, $ij = -ji = k$. Thus we only need to check if these hold for the corresponding elements in $A/A\cdot(x^2+1)$. $i^2 = -1$ is true in $A/A\cdot(x^2+1)$. $x^2 = x^2 - (x^2+1) = -1$ thus the identity corresponding to $j^2 = -1$ is true in $A/A\cdot(x^2+1)$. Of course $i \cdot x = ix$ in $A/A\cdot(x^2+1)$, corresponding to $ij = k$ in $\mathbb{H}$. Finally $xi = \bar{i}x = -ix$ in $A/A\cdot(x^2+1)$ which corresponds to $ji = -ij$ in $\mathbb{H}$. This proves that $A/A\cdot(x^2+1)$ is isomorphic to $\mathbb{H}$.