1.14
Homework 1

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Ex. 1.14. *Suppose an element a in a ring has some right inverse b but no left inverse. Then a has infinitely many right inverses.*

Proof. Consider the elements $(ba-1)a^k$ for k a nonnegative integer. Since a has no left inverse, $ba-1$ cannot equal zero. This forces $(ba-1)a^k$ to be nonzero as well since $(ba-1)a^kb^k = ba-1 \neq 0$ but $0 \cdot b^k = 0$. However, $(ba-a)b = bab-b = b \cdot 1 - b = 0$. Hence $(ba-1)a^kb^l$ is zero if and only if $l > k$. This forces the $(ba-1)a^k$ to be distinct, as the power of b which makes $(ba-1)a^k$ by right multiplication depends uniquely on k. Note $a(ba-1)a^k = (aba-a)a^k = (1 \cdot a - a)a^k = 0$. Adding b and left multiplying by a, we find $a((ba-1)a^k+b) = 0+ab = 1$. Hence elements of the form $(ba-1)a^k+b$ are right inverses of a. Finally we note that these inverses are distinct as otherwise $(ba-1)a^k+b = (ba-1)a^l+b$, hence $(ba-1)a^k = (ba-1)a^l$, which is impossible unless $k = l$. Thus we have an infinite set of right inverses of a.

□

Corollary. *Any element a of some finite ring which has some right inverse b must also have that $ba = 1$.*

Proof. Since the ring is finite, a cannot have infinitely many right inverses, hence it must also have a left inverse c. Then $ca = ab = 1$. Then $c = c \cdot 1 = cab = 1 \cdot b = b$. Hence $ba = 1$. □