Problem. *Let M be a Noetherian left module over a ring R. Show that any surjective R-endomorphism of M is an automorphism. Using this, show that any left Noetherian ring R is Dedekind-finite.*

Solution. Let M be a Noetherian left module over a ring R, and let $\phi \in \mathrm{End}_R(M)$ be surjective. Then the chain

$$\ker\phi \subseteq \ker\phi^2 \subseteq \ker\phi^3 \subseteq \cdots$$

must eventually become constant, so for some k we must have

$$\ker\phi^k = \ker\phi^{k+1}.$$

Let $a \in \ker\phi$. Then since ϕ is surjective there exists $b \in M$ such that $\phi^k(b) = a$. But then

$$\phi^{k+1}(b) = \phi\left(\phi^k(b)\right) = \phi(a) = 0,$$

so $b \in \ker\phi^{k+1} = \ker\phi^k$, thus

$$a = \phi^k(b) = 0,$$

so $\phi \in \mathrm{Aut}_R(M)$.

Now let R be a left Noetherian ring, and let $a, b \in R$ such that $ab = 1$. Define

$$\phi : R \to R : r \mapsto rb.$$

We clearly see that $\phi \in \mathrm{End}_R(R)$, and since

$$\phi(a) = ab = 1$$

we see that ϕ is surjective. Then by the above discussion, $\phi \in \mathrm{Aut}_R(R)$. Then

$$\phi(ba - 1) = (ba - 1)b = (ba)b - b = b(ab) - b = 0,$$

thus $ba = 1$, so R is Dedekind-finite.

$\square$