The Principle of Inclusion and Exclusion.

Version 1. The principle of incl./exlc. counts the size of a union.

$$|A_1 \cup \cdots \cup A_n| = \sum_{1\leq i\leq n} |A_i| - \sum_{1\leq i<j\leq n} |A_i \cap A_j| + \cdots + (-1)^{n+1}|A_1 \cap \cdots \cap A_n|.$$

Version 2. Let X be a set and let $\mathcal{P}$ be a set of properties the elements of X may have. If $N_=(S)$ is the number of elements of X that have exactly the properties in $S \subseteq \mathcal{P}$ and $N_\geq(S)$ is the number of elements of X that have at least the properties in $S \subseteq \mathcal{P}$, then

$$\begin{aligned} N_\geq(S) &= \textstyle\sum_{S\subseteq T\subseteq \mathcal{P}} N_=(T) \quad \text{and} \\ N_=(S) &= \textstyle\sum_{S\subseteq T\subseteq \mathcal{P}} (-1)^{|T|-|S|} N_\geq(T). \end{aligned}$$

The first formula is trivial; the principle of inclusion and exclusion is the second formula.

Exercises.

(1) How many positive integers less than 1000 are not divisible by 2, 3, 5 or 7?

Hint: Use properties P_2, P_3, P_5 and P_7, where n has P_i exactly when n is divisible by i.

$N_=(\emptyset) = 1000-500-333-200-142+166+100+71+66+47+28-33-23-14-9+4 = 228.$

(2) How many positive integers less than 250 are not perfect powers?

It suffices to count the elements of $\{2, 3, \ldots, 250\}$ that are not squares, cubes, 5th powers or 7th powers. This is similar to the previous problem. The answer is 226.

(3) How many 5-digit numbers fail to contain the sequence 01? How about 00?

First part:

$$10^5 - 4 \cdot 10^3 + 3 \cdot 10 = 96030.$$

Second part:

$$10^5 - 4 \cdot 10^3 + 3 \cdot 10^2 + 3 \cdot 10 - 22 + 1 = 96309.$$

(This assumes that 5-digit numbers may have leading 0's. If you assume that they do not, then the answers are: 87309 (first part), 87480 (second part).)

(4) How many 6-digit numbers have the property that, for every k, the kth digit is different than the $(7-k)$th digit?

Hint: Use properties P_1, P_2, P_3, where n has P_k exactly when the kth digit equals the $(7-k)$th digit.

$$\binom{3}{0}10^6 - \binom{3}{1}10^5 + \binom{3}{2}10^4 - \binom{3}{3}10^3 = 10^3 9^3 = 729000.$$

(This assumes that 6-digit numbers may have leading 0's. If you assume that they do not, then the answer is: $10^2 9^4 = 656100$.