The Principle of Inclusion and Exclusion.

Version 1. The principle of incl./exlc. counts the size of a union.

$$|A_1 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|.$$

Version 2. Let X be a set and let $\mathcal{P}$ be a set of properties the elements of X may have. If $N_=(S)$ is the number of elements of X that have exactly the properties in $S \subseteq \mathcal{P}$ and $N_\geq(S)$ is the number of elements of X that have at least the properties in $S \subseteq \mathcal{P}$, then

$$\begin{aligned} N_\geq(S) &= \textstyle\sum_{S \subseteq T \subseteq \mathcal{P}} N_=(T) \quad \text{and} \\ N_=(S) &= \textstyle\sum_{S \subseteq T \subseteq \mathcal{P}} (-1)^{|T|-|S|} N_\geq(T). \end{aligned}$$

The first formula is trivial; the principle of inclusion and exclusion is the second formula.

Exercises.

(1) How many positive integers less than 1000 are not divisible by 2, 3, 5 or 7?

(2) How many positive integers less than 250 are not perfect powers?

(3) How many 5 digit numbers fail to contain the sequence 01? How about 00?

(4) How many 6 digit numbers have the property that, for every k, the kth digit is different than the $(7-k)$th digit?