## Graph theory glossary

O. Notation: If $V$ is a set, then $V^{(2)}$ is the set of all ***doubletons*** of $V$; i.e., the set of all 2-element subsets of $V$.

I. Basic definitions

(a) A ***graph*** is an ordered pair $(V, E)$ where $V$ is a set and $E$ is a subset of $V^{(2)}$. Elements of $V$ are called ***vertices*** and elements of $E$ are called ***edges***. (Alternatively, a graph is an ordered pair $G = (V, E)$ where $V$ is a set and $E$ is a symmetric, irreflexive, binary relation on $V$.)

(b) Vertices $u, v \in V$ are ***adjacent*** in $G$ if $\{u, v\} \in E$. We also express adjacency by saying that $u$ and $v$ are ***neighbors*** in $G$. The adjacency relation is recorded in the ***adjacency matrix***, which is the $|V| \times |V|$ matrix $[a_{uv}]$ where $a_{uv} = 1$ if $u$ is adjacent to $v$ and $a_{uv} = 0$ if $u$ is not adjacent to $v$.

(c) A vertex $v \in V$ and an edge $\{a, b\} \in E$ are ***incident (to each other)*** if $v \in \{a, b\}$; i.e., if $v = a$ or $v = b$. The incidence relation is recorded in the ***incidence matrix***, which is the $|V| \times |E|$ matrix $[a_{ve}]$ where $a_{ve} = 1$ if $v$ is incident to $e$ and $a_{ve} = 0$ if $u$ is not incident to $v$.

II. Comparison

(a) An ***isomorphism*** from a graph $G = (V, E)$ to a graph $G' = (V', E')$ is a bijection $f\colon V \to V'$ that ***preserves and reflects*** edges. "Preserves and reflects" means that $\{u, v\} \in E$ if and only if $\{f(u), f(v)\} \in E'$. $G$ and $G'$ are ***isomorphic*** if there is an isomorphism from one to the other. An isomorphism from $G$ to itself is an ***automorphism*** of $G$.

(b) A ***subgraph*** of $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. An ***induced subgraph*** of $G = (V, E)$ is a subgraph $G' = (V', E')$ where $E' = E \cap (V')^{(2)}$.

(c) The ***complement*** of $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$ that has the same vertex set as $G$, but whose edge set consists of all edges *not* in the edge set of $G$.

III. Connectivity

(a) A ***walk*** in $G$ is a sequence $(v_0, v_1, \ldots, v_\ell)$ of vertices such that $\{v_i, v_{i+1}\} \in E$ for all $i$. The ***length*** of the walk $(v_0, v_1, \ldots, v_\ell)$ is $\ell$.

(b) A ***trail*** is a walk with no repeated edges. An ***Eulerian trail*** is a trail involving every edge exactly once.

(c) A walk or trail is ***closed*** if it starts and ends at the same vertex. Otherwise it is ***open***.

(d) A trail with no repeated vertices or edges is a ***path***. A closed trail with no repeated vertices or edges except at the start and end is a ***cycle***. (Possible ambiguity: is the trail of length zero called a path or a cycle? Convention says: <u>path</u>. As a consequence, any cycle has length at least three.)

(e) A ***Hamiltonian path (cycle)*** is a path (cycle) involving every vertex exactly once (except at the start end end).
(f) Vertex $u$ is ***connected to*** vertex $v$ if there is a path $(u, \dots, v)$ starting at $u$ and ending at $v$. (We allow $u = v$ in this definition.)
(g) The ***connected component*** of $v$ is the set of vertices connected to $v$.
(h) $G$ is ***connected*** if any two vertices of $G$ are connected.

IV. Special subsets
(a) A ***clique*** in $G = (V, E)$ is a subset $C \subseteq V$ such that any two vertices in $C$ are adjacent in $G$.
(b) An ***independent set*** in $G = (V, E)$ is a subset $I \subseteq V$ such that no two vertices in $I$ are adjacent in $G$.

V. Size measures
(a) The ***order*** of a graph $G = (V, E)$ is $|V|$.
(b) The ***size*** of a graph $G = (V, E)$ is $|E|$.
(c) The ***degree*** of a vertex $v \in V$ is the number of edges incident to $v$.
(d) $\Delta(G)$ denotes the maximum degree of a vertex of $G$, and $\delta(G)$ denotes the minimum degree of a vertex of $G$.
(e) The ***distance*** between $u, v \in V$, written $d(u, v)$, is the length of the shortest path between $u$ and $v$.
(f) The ***diameter*** of $G$ is the supremum of the distances between any two vertices.
(g) The ***independence number*** of $G$ is the size of the largest independent set $G$, written $\alpha(G)$.
(h) The ***clique number*** of $G$ is the size of the largest clique in $G$, written $\omega(G)$.