DISCRETE MATH
MIDTERM Z

Name:______________________________

You have 50 minutes for this exam. If you have a question, raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

1. Define the italicized boldface words or phrases.

(a) A is a ***subset*** of B:

For all x, $(x \in A)$ implies $(x \in B)$.

(b) The ***natural numbers***:

The ***natural numbers*** is the set consisting of the elements common to all inductive sets.

2.

(a) One axiom of set theory is

$$\exists x\, \forall y\, (\neg(y \in x)).$$

What is the name of this axiom, and how would you write it in English?

This is the Axiom of the Empty Set. In English, "there is a set with no elements".

(b) Write a formal sentence expressing "There is no set of all sets."

$$\neg(\exists x\, (\forall y\, (y \in x))).$$

3. Give a winning strategy for some quantifier to decide the truth of

$$(\exists x > 0)(\forall y > 0)\ (y + y > x)$$

(a) in the real numbers, $(\mathbb{R}; +, >, 0)$.

A winning strategy for $\forall$ is to choose $y = x/2$.

(b) in the natural numbers, $(\mathbb{N}; +, >, 0)$.

A winning strategy for $\exists$ is to choose $x = 1$.

4. Show that $(m \cdot n)^p = m^p \cdot n^p$ for any $m, n, p \in \mathbb{N}$. You may assume the truth of any valid laws of arithmetic that involve *addition or multiplication only* (not exponentiation). Explain your steps.

(Basis of Induction, $p = 0$.)

$$\begin{aligned} (m \cdot n)^0 &= 1 && ((\mathrm{IC}), \exp) \\ &= 1 \cdot 1 && (\text{Lemma. } 1 \cdot 1 = 1.) \\ &= m^0 \cdot n^0. && ((\mathrm{IC}), \exp) \end{aligned}$$

(Inductive Step.) Assume that $(m \cdot n)^p = m^p \cdot n^p$ for any $m, n \in \mathbb{N}$ and some fixed $p \in \mathbb{N}$.

$$\begin{aligned} (m \cdot n)^{S(p)} &= (m \cdot n)^p \cdot (m \cdot n) && ((\mathrm{RR}), \exp) \\ &= (m^p \cdot n^p) \cdot (m \cdot n) && (\text{Inductive Hypothesis}) \\ &= (m^p \cdot n^p) \cdot (n \cdot m) && (\text{Commutative Law for } \cdot) \\ &= (m^p \cdot ((n^p \cdot n) \cdot m)) && (\text{Associative Law for } \cdot\text{, twice}) \\ &= (m^p \cdot (n^{S(p)} \cdot m)) && ((\mathrm{RR}), \exp) \\ &= (m^p \cdot m) \cdot n^{S(p)} && (\text{Comm.+Assoc. Laws for } \cdot) \\ &= m^{S(p)} \cdot n^{S(p)} && ((\mathrm{RR}), \exp) \end{aligned}$$

5. Suppose that all students take 6 courses apiece, and that there are 12 possible final examination times. How many different examination schedules are possible? (A typical final examination schedule would look like this: "Final for course 1 is on Wednesday at 7:30am; final for course 2 is on Monday at 1:30pm; etc.")

$(12)_6$. (Reason: Schedules may be thought of as injective functions from the set $\{\text{course } 1, \text{course } 2, \ldots, \text{course } 6\}$ to the set of examination times.)