

DISCRETE MATH  
MIDTERM Y

Name: \_\_\_\_\_

You have 50 minutes for this exam. If you have a question, raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

1. Define the italicized boldface words.

(a) “ $A$  is a ***subset*** of  $B$ ”.

For all  $x$ ,  $(x \in A)$  implies  $(x \in B)$ .

(b) “ $f$  is a ***function*** from  $A$  to  $B$ ”.

$F$  is a ***function*** from  $A$  to  $B$  if  $F$  is a binary relation from  $A$  to  $B$  that satisfies the *function rule*.

2.

(a) One axiom of set theory is

$$\forall x \forall y \exists z \forall w ((w \in z) \leftrightarrow ((w = x) \vee (w = y)))$$

What is the name of this axiom, and how would you write it in English?

This is the Axiom of Pairing. In English, “for any two sets  $x$  and  $y$  there is a set  $z$  whose only elements are  $x$  and  $y$ ”.

(b) Write a formal sentence expressing “There is no set of all sets.”

$$\neg(\exists x (\forall y (y \in x))).$$

3. Give a winning strategy for some quantifier to decide the truth of

$$(\forall x > 0)(\exists y > 0) (x = y^2)$$

- (a) in the real numbers,  $(\mathbb{R}; \cdot, >, 0)$ .

A winning strategy for  $\exists$  is to choose  $y = \sqrt{x}$  if  $x > 0$ , and to choose  $y = 1$  if  $\forall$  chose  $x \leq 0$ . (If  $\forall$  chooses  $x \leq 0$ , then  $\exists$  has already won, so  $\exists$  can choose an arbitrary value for  $y$ .)

- (b) in the natural numbers,  $(\mathbb{N}; \cdot, >, 0)$ .

A winning strategy for  $\forall$  is to choose  $x = 2$ .

4. Show that  $(m \cdot n)^p = m^p \cdot n^p$  for any  $m, n, p \in \mathbb{N}$ . You may assume the truth of any valid laws of arithmetic that involve *addition or multiplication only* (not exponentiation). Explain your steps.

(Basis of Induction,  $p = 0$ .)

$$\begin{aligned} (m \cdot n)^0 &= 1 && ((IC), \exp) \\ &= 1 \cdot 1 && (\text{Lemma. } 1 \cdot 1 = 1.) \\ &= m^0 \cdot n^0. && ((IC), \exp) \end{aligned}$$

(Inductive Step.) Assume that  $(m \cdot n)^p = m^p \cdot n^p$  for any  $m, n \in \mathbb{N}$  and some fixed  $p \in \mathbb{N}$ .

$$\begin{aligned} (m \cdot n)^{S(p)} &= (m \cdot n)^p \cdot (m \cdot n) && ((RR), \exp) \\ &= (m^p \cdot n^p) \cdot (m \cdot n) && (\text{Inductive Hypothesis}) \\ &= (m^p \cdot n^p) \cdot (n \cdot m) && (\text{Commutative Law for } \cdot) \\ &= (m^p \cdot ((n^p \cdot n) \cdot m)) && (\text{Associative Law for } \cdot, \text{ twice}) \\ &= (m^p \cdot (n^{S(p)} \cdot m)) && ((RR), \exp) \\ &= (m^p \cdot m) \cdot n^{S(p)} && (\text{Comm.+Assoc. Laws for } \cdot) \\ &= m^{S(p)} \cdot n^{S(p)} && ((RR), \exp) \end{aligned}$$

5. How many ways can students line up for ice cream if there are  $m$  girls,  $n$  boys, and all the girls are required to line up ahead of all the boys?

$$m! \cdot n!$$