DISCRETE MATH
MIDTERM X

Name:____________________________

You have 50 minutes for this exam. If you have a question, raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

1. Define the italicized boldface words or phrases.

(a) The ***natural numbers***:

The ***natural numbers*** is the set consisting of the elements common to all inductive sets.

(b) ***function***:

F is a ***function*** from A to B if F is a binary relation from A to B that satisfies the *function rule.*

2.

(a) One axiom of set theory is

$$\exists x \, \forall y \, (\neg(y \in x)).$$

What is the name of this axiom, and how would you write it in English?

This is the Axiom of the Empty Set. In English, "there is a set with no elements".

(b) Write a formal sentence expressing "There is no set of all sets."

$$\neg(\exists x \, (\forall y \, (y \in x))).$$

3. Give a winning strategy for some quantifier to decide the truth of

$$(\forall x > 0)(\exists y > 0)\ (x > y^2)$$

(a) in the real numbers, $(\mathbb{R}; \cdot, >)$.

A winning strategy for $\exists$ is to choose y so that $0 < y < \sqrt{x}$ if $\forall$ chose $x > 0$, and to choose $y = 1$ if $\forall$ chose $x \leq 0$. (If $\forall$ chooses $x \leq 0$, then $\exists$ has already won, so $\exists$ can choose an arbitrary value for y.)

(b) in the natural numbers, $(\mathbb{N}; \cdot, >)$.

A winning strategy for $\forall$ is to choose $x = 1$.

4. Show that $(m \cdot n)^p = m^p \cdot n^p$ for any $m, n, p \in \mathbb{N}$. You may assume the truth of any valid laws of arithmetic that involve *addition or multiplication only* (not exponentiation). Explain your steps.

(Basis of Induction, $p = 0$.)

$$\begin{aligned} (m \cdot n)^0 &= 1 && \text{((IC), exp)} \\ &= 1 \cdot 1 && \text{(Lemma. } 1 \cdot 1 = 1.\text{)} \\ &= m^0 \cdot n^0. && \text{((IC), exp)} \end{aligned}$$

(Inductive Step.) Assume that $(m \cdot n)^p = m^p \cdot n^p$ for any $m, n \in \mathbb{N}$ and some fixed $p \in \mathbb{N}$.

$$\begin{aligned} (m \cdot n)^{S(p)} &= (m \cdot n)^p \cdot (m \cdot n) && \text{((RR), exp)} \\ &= (m^p \cdot n^p) \cdot (m \cdot n) && \text{(Inductive Hypothesis)} \\ &= (m^p \cdot n^p) \cdot (n \cdot m) && \text{(Commutative Law for } \cdot\text{)} \\ &= (m^p \cdot ((n^p \cdot n) \cdot m)) && \text{(Associative Law for } \cdot\text{, twice)} \\ &= (m^p \cdot (n^{S(p)} \cdot m)) && \text{((RR), exp)} \\ &= (m^p \cdot m) \cdot n^{S(p)} && \text{(Comm.+Assoc. Laws for } \cdot\text{)} \\ &= m^{S(p)} \cdot n^{S(p)} && \text{((RR), exp)} \end{aligned}$$

5. Suppose there are four candidates running for office.

(a) In how many ways can 100 distinct voters cast their votes if each voter must vote for exactly one candidate?

4^{100}. (Reason: A voting scheme is just a function from the set of voters to the set of candidates.)

(b) Now suppose that each voter must vote for two candidates instead of only one. In how many ways can the 100 voters cast their votes?

$\binom{4}{2}^{100} = 6^{100}$. (Reason: Now we count functions from the set of voters to the set of 2-element subsets of candidates.)