DISCRETE MATH (MATH 2001)

REVIEW SHEET II

VI. Counting, Part 2
  (a) Binomial coefficients.
    (i) There are $\binom{n}{k}$ k-elements subsets of an n-element set.
    (ii) Binomial Theorem.
    (iii) Pascal's identity. Pascal's triangle: unimodality of nth row and nth row sum.
    (iv) Generalization to multinomial coefficients: there are $\binom{n}{k_1,\ldots,k_r}$ ordered partitions of n into cells of sizes $(k_1, \ldots, k_r)$; Multinomial Theorem; Pascal's Pyramid; nth row sum.
  (b) Counting multisets.
  (c) Principle of Inclusion and Exclusion: Formula. Counting surjective functions. Stirling numbers of the second kind.
  (d) Distribution problems.

VII. The Canonical Factorization of a Function
  (a) Kernel. Equivalence relation.
  (b) Coimage. Partition.
  (c) Natural function. Inclusion function. Induced function. $f = \mathrm{inc} \circ \overline{f} \circ \mathrm{nat}$.
  (d) Well-defined functions.

VIII. The Construction of $\mathbb{Z}$.
  (a) Definition $\mathbb{Z}$. Arithmetic of $\mathbb{Z}$.

IX. Graph Theory
  (a) Graph theory terminology.
  (b) Examples: paths, cycles, complete graphs, complete r-partite graphs, Petersen graph.
  (c) Eulerian trails. Euler's Theorem.
  (d) Hamiltonian paths and cycles. Ore's Theorem.
  (e) Graph coloring.
    (i) Four color theorem.
    (ii) Chromatic number (definition, and estimate $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$)
    (iii) G is 2-colorable iff G has no odd cycles.
  (f) Planar graphs
    (i) Euler's Formula: $v - e + r = 2$.
    (ii) Edge bounds for loopless planar graphs with enough vertices ($e \leq 3v - 6$ if $3 \leq v$; $e \leq 2v - 4$ if $4 \leq v$ and graph is bipartite).
    (iii) Kuratowski's characterization of planar graphs.
    (iv) Euler characteristic of a surface.

General advice on preparing for a math test.

Be prepared to demonstrate understanding in the following ways.

(1) Know the definitions of new concepts, and the meanings of the definitions.
(2) Know the statements and meanings of the axioms and major theorems.
(3) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
(4) Know how to perform the different kinds of calculations discussed in class.
(5) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
(6) Know how to correct mistakes made on old HW.

More specific advice.

Be prepared to demonstrate understanding in the following ways.

(1) Know the definitions of the following: binomial and multinomial coefficients; multiset; (ordered or unordered) partition; equivalence relation; kernel; image; coimage; natural function; inclusion function; induced function; $\langle \mathbb{Z}; \cdot, +, -, 0, 1\rangle$; definitions on the graph theory handout.
(2) Know the statements and meanings of: Binomial and Multinomial Theorems; Principle of Inclusion and Exclusion; Euler's Theorem (about Eulerian trails); Ore's Theorem; The Four Color Theorem; Euler's Formula; Kuratowski's Theorem; The Classification of Surfaces.
(4) Know how to: apply all the counting formulas on the 'Distributions' handout; describe the canonical factorization of a function; determine if a function is well-defined; prove laws of integer arithmetic; find an Eulerian trail or prove that none exists; determine if a graph is 2-colorable; determine the Eulewr characteristic of a surface; write formal sentences expressing graph theoretical statements.

Test your understanding.

(1) How many ways are there to make a circular necklace with n beads of different colors if two necklaces are considered to be the same if they differ by a rotation? What if two necklaces are considered to be the same if they differ by a rotation or a flip?

(2) What is the constant term in $(x^{-2} + 2x^{-1} + 3 + 5x)^3$?

(3) You have just given birth to octuplets. How many ways can you name your children if you only like the names Billy Bob, Jim Bob and Sue Bob?

(4) If you deal a random 2-card hand, what is the probability of blackjack? (An ace together with a 10 or face card.)

(5) Describe the procedure for constructing the set of integers from the set of natural numbers. Without proving anything, identify the statement that must be proved to verify that the procedure works.

(6) Show that the integers satisfy $\forall x\, \forall y\, (x + y = y + x)$.

(7) If $x = [(k, \ell)]_E$ and $y = [(m, n)]_E$, then set $x * y = [(k \cdot m, \ell \cdot n)]_E$. Is this a well-defined operation on $\mathbb{Z}$?

(8) How many loopless multigraphs with vertex set $\{v_1, \ldots, v_n\}$ have k edges? What if loops are allowed?

(9) Let C_n be a cycle of length n and let K be a set of k colors. How many proper colorings of C_n are there which use only colors from K?

(10) Consider a graph to be a structure $G = \langle V; E \rangle$ where E is a binary predicate on the set V. Thus $E(a, b)$ holds if vertices a and b are connected by an edge. Write formal sentences that hold in G iff
  (a) any two vertices are connected by a path of length 3.
  (b) K_4 is a subgraph.
  (c) the diameter is 2.

(11) Write a formal sentence that distinguishes between the Petersen graph and K_5.

(12) Does the Petersen graph have a Hamiltonian cycle?

(13) Give a simple description of the class of graphs that satisfy the following sentence.

$$\forall x\, \exists y\, \forall u\, \exists v\, (E(x, y) \wedge E(y, u) \wedge E(u, v) \wedge E(v, x))$$

(14) Find the Euler characteristic of the 2-holed torus.