DISCRETE MATH (MATH 2001)

REVIEW SHEET I

I. Set Theory
   (a) Informal notion of a set. The axioms.
   (b) Valid constructions of new sets (pairing, union, power set, comprehension, intersection)
   (c) Empty set, successor of a set.
   (d) Inductive sets, natural numbers.
   (e) Russell's Paradox.

II. Logic
   (a) Formulas
      (i) Symbols: variables, equality, logical connectives, quantifiers, predicate symbols, punctuation symbols.
      (ii) Atomic formulas, formulas and sentences.
      (iii) Formula trees.
   (b) Propositional logic
      (i) Truth tables.
      (ii) Tautologies, contradictions, logical equivalence.
      (iii) Contrapositive and converse.
      (iv) Equivalence of $(H \to C)$, $((\neg C) \to (\neg H))$, and $((H \wedge (\neg C)) \to \text{False})$. Methods of proof.
   (c) Structures (definition and examples).
   (d) Truth of a sentence in a structure.
      (i) Converting a sentence to prenex form.
      (ii) Quantifier games to determine the truth of a sentence in prenex form in a given structure.

III. Induction
   (a) Ordinary induction.
   (b) Strong induction.
   (c) Recursive definitions of arithmetic operations on $\mathbb{N}$: $x + y, xy, x^y$.
   (d) Use of induction to prove laws of arithmetic.

IV. Relations
   (a) Ordered pairs, triples and n-tuples. $A \times B$.
   (b) Definition of a function. Representations of functions.
   (c) Injections, surjections, bijections.

V. Counting
   (a) Definitions of $|A| = |B|$, $|A| = m$, finite and infinite.
   (b) Sum Rule and Product Rule.

(c) Simple counting formulas:
  (i) There are m^n functions from an n-element set to an m-element set.
  (ii) There are $(m)_n = m \cdot (m-1)\cdots(m-n+1)$ injective functions from an n-element set to an m-element set. There are $n!$ bijections between two n-element sets.
  (iii) There are $n!$ ways to linearly order an n-element set.

General advice on preparing for a math test.
Be prepared to demonstrate understanding in the following ways.

(1) Know the definitions of new concepts, and the meanings of the definitions.
(2) Know the statements and meanings of the axioms and major theorems.
(3) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
(4) Know how to perform the different kinds of calculations discussed in class.
(5) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
(6) Know how to correct mistakes made on old HW.

More specific advice.
Be prepared to demonstrate understanding in the following ways.

(1) Know the definitions of the following: empty set; successor; subset; power set; union; intersection; unordered pair; ordered pair; $A\times B$; inductive set; natural numbers; logical connectives; propositional formula; truth table; tautology; contradiction; logical equivalence; contrapositive; converse; relation; predicate; atomic formulas; formula; formula tree; prenex form; arithmetic operations on $\mathbb{N}$; function; injection; surjection; bijection; $|A| = |B|$; $|A| = n$; finite; infinite.
(2) Know the statements and meanings of: the axiom of the empty set; the axiom of extensionality; the axiom of pairing; the axiom of union; the axiom of power set; the axiom of comprehension; Russell's Paradox; the theorem proving that induction works; Sum Rule; Product Rule.
(4) Know how to: prove two sets are equal; organize a proof so that it is a direct proof, a proof of the contrapositive, or a proof by contradiction; rewrite English sentences as formal sentences and rewrite formal sentences as English sentences; create a formula tree; put a formula in prenex form; test a sentence for truth in a structure; prove statements by induction; establish that $|A| = |B|$ or $|A| \neq |B|$; count the number of all functions, injective functions or bijective functions from a set of size m to a set of size n.

Test your understanding.

(1) Explain why $2 + 2 = 4$.

(2) Show that $A \subseteq B$ if and only if $A \cup B = B$.

(3) Explain why induction is a valid form of proof.

(4) Prove that $m^{n+p} = m^n m^p$ for all $m, n, p \in \mathbb{N}$.

(5) Write a formal sentence expressing the axiom of union. Then draw a formula tree for your sentence.

(6) Suppose you want to prove a theorem with two hypotheses: $((H_1 \wedge H_2) \rightarrow C)$. Which of the following proof strategies would <u>suffice</u> to prove the theorem? Explain your answer.
 (i) A proof of $(((\neg C) \wedge H_2) \rightarrow (\neg H_1))$ would suffice.
 (ii) A proof of $((\neg H_1) \vee (\neg H_2) \vee C)$ would suffice.
 (iii) A proof of $((\neg C) \rightarrow ((\neg H_1) \wedge (\neg H_2)))$ would suffice.

(7) Describe a winning strategy for either $\exists$ or $\forall$, which determines the truth of

$$\forall x\ (\exists y\ (x = y^2) \rightarrow \exists z\ (x + 1 = z^2))$$

in (i) $\langle \mathbb{R}; +, \cdot, 1 \rangle$, (ii) $\langle \mathbb{Z}; +, \cdot, 1 \rangle$.

(8) Put $(A \leftrightarrow \forall x\ B(x))$ in prenex form. You may assume that A has no free variables.

(9) What is a function? (Give the definition.) If f is a function, under what circumstances will f^{-1} also be a function?

(10) If $U \subseteq A$, then the *characteristic function* of U is the function $\chi_U \colon A \rightarrow \{0, 1\}$ defined by $\chi_U(a) = 1$ if $a \in U$ and $\chi_U(a) = 0$ if $a \notin U$.
 (i) Explain why every function from A to $\{0, 1\}$ is the characteristic function of some subset $U \subseteq A$.
 (ii) Show that if K is the set of all characteristic functions defined on A, then $|K| = |\mathcal{P}(A)|$.
 (iii) Explain why $|\mathcal{P}(A)| = 2^{|A|}$.