Map Coloring

Discrete Math

Math 2001

Spring 2011

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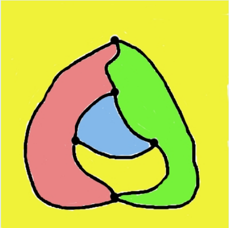

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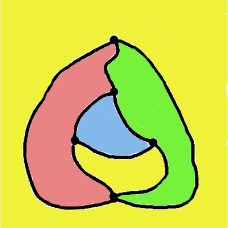

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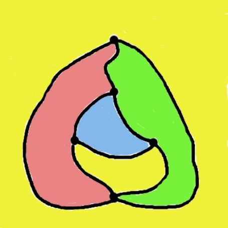

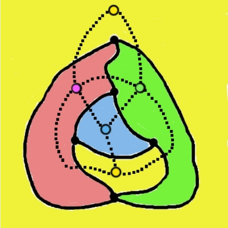

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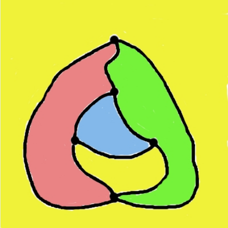

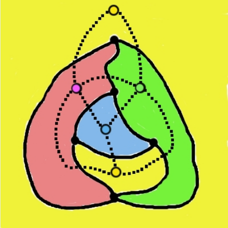

If a graph represents a planar map, then it is possible to color its vertices so that no two adjacent vertices have the same color.

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Map coloring research may be split into two (mostly unrelated) areas:

1. Graph coloring.
2. Drawing graphs on surfaces.

Graph Coloring

Definition

If $G = (V, E)$ is a graph and C is a color set, then a ***proper coloring*** of G is a function $f : V \to C$ such that $\forall u, v(E(u, v) \to (f(u) \neq f(v)))$.

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2. *In fact,* $\omega(G) \leq \chi(G)$. *(Use* $H \leq G \implies \chi(H) \leq \chi(G)$*.)*
3. *On the other hand,* $\chi(G) \leq \Delta(G) + 1$, *where* $\Delta(G)$ *denotes the largest degree of a vertex of* G.

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- Solve Sudoku.

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This conflicts slightly with the notation K_n.

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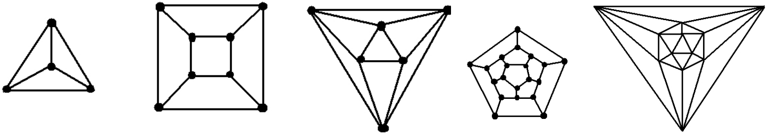
This suggests the idea that one might be able to distinguish between surfaces (like the sphere versus the torus) by determining which graphs represent dissections of the surfaces into polygons.

Numerical Invariants of Planar Drawings



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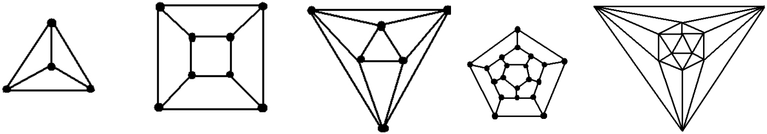

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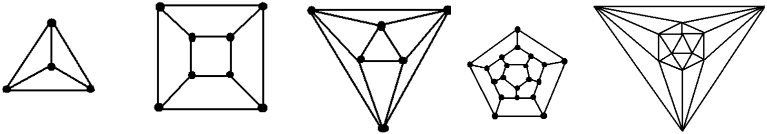

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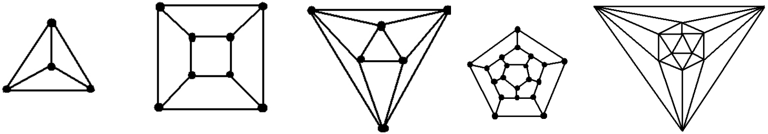

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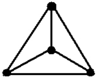 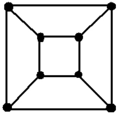 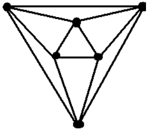 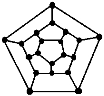 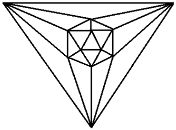

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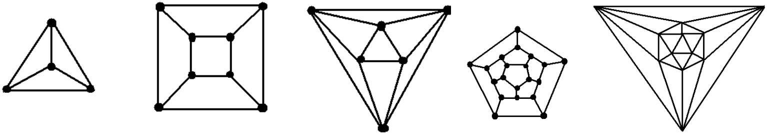

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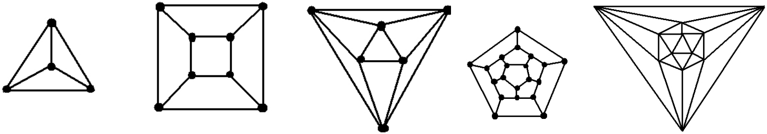

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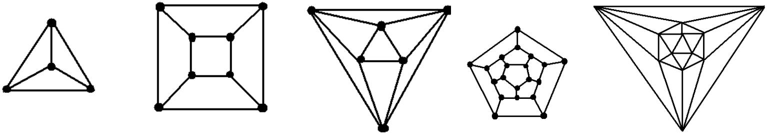

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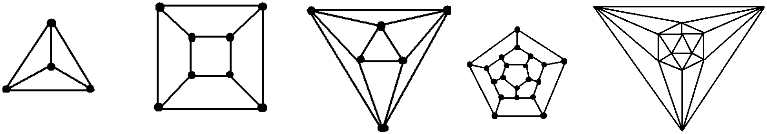

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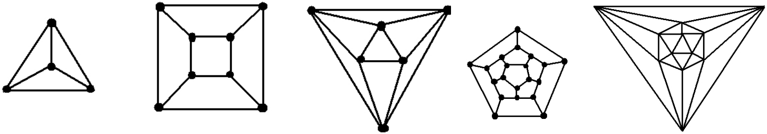

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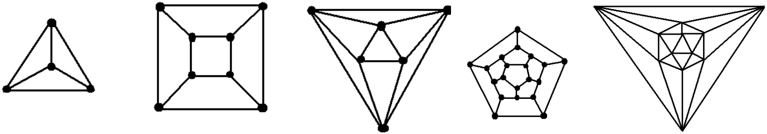

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The proof actually shows that for *any* surface S, the number $v - e + r$ is the same for all graphs that represent dissections of S into polygons. This number may be different for different surfaces. It is called the ***Euler characteristic*** of S, $\chi(S)$.

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2. *If G is bipartite, then* $e \leq 2v - 4$.

Corollary

K_5 and $K_{3,3}$ are not planar.

Kuratowski's Theorem

A graph G is planar iff it has no subdivision of K_5 or $K_{3,3}$ as a subgraph.