Complexes and Exact Sequences in Abelian Categories

Homological Algebra

Feb 10, 2010

Subobjects

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Proof. $\mu = [\ker(\operatorname{coker}(\varphi))]$ has this property. There is a unique ε such that

$$A \xrightarrow{\ \varepsilon\ } K \xrightarrow{\ \mu\ } B \xrightarrow{\ \gamma\ } C$$

and $\mu\varepsilon = \varphi$, so φ factors through μ. HW2.5(a) shows that ε is epi.

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Now suppose that φ factors through some other monomorphism

$$A \xrightarrow{\ \varepsilon'\ } K' \xrightarrow{\ \mu'\ } B.$$

By factoring ε if necessary, we may assume that it is epi. Now the HW2.5(b) proves that $\mu \leq \mu'$. $\square$

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Cor. (HW2.6(b))
If φ is a monomorphism, then $\mathrm{im}(\varphi) = [\varphi]$.

Complexes and exact sequences

A sequence

$$\cdots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

where $d_n d_{n+1} = 0$ for all n is a *complex*.

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Exact sequences are complexes: recall that $\mathrm{im}(d_{n+1})$ is the smallest subobject of C_n such that d_{n+1} factors as $\cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{\varepsilon} I = K \xrightarrow{\mu} C_n \xrightarrow{d_n} \cdots$ with μ representing $\mathrm{im}(d_{n+1})$.

Facts about exactness

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- $0 \to A \to B \to C \to 0$ is exact iff $A \to B$ is a monomorphism and $B \to C$ is its cokernel.

Pullbacks

A diagram

$$\begin{array}{ccc} P & \xrightarrow{\pi_B} & B \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\alpha} & C \end{array}$$

is a *pullback square* if for every pair of maps $\mu\colon X \to A$ and $\nu\colon X \to B$ such that

$$\begin{array}{ccc} X & \xrightarrow{\nu} & B \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\alpha} & C \end{array}$$

commutes there is a unique $\lambda\colon X \to P$ such that $\mu = \pi_A\lambda$ and $\nu = \pi_B\lambda$. (P, π_A, π_B) is the *pullback* of α and β. *Pushouts* are defined dually.

Converting squares to sequences

From the (possibly noncommutative) diagram

$$\begin{tikzcd} P \arrow[r, "\delta"] \arrow[d, "\gamma"'] & B \arrow[d, "\beta"'] \\ A \arrow[r, "\alpha"] & C \end{tikzcd}$$

we can construct the sequence $P \xrightarrow{i_A\gamma + i_B\delta} A \oplus B \xrightarrow{\alpha\pi_A - \beta\pi_B} C$. The composition of the maps in the sequence is $\alpha\gamma - \beta\delta$, so the diagram commutes iff the sequence is a complex. Conversely, given $P \xrightarrow{\sigma} A \oplus B \xrightarrow{\tau} C$, we can construct

$$\begin{tikzcd} P \arrow[r, "\pi_B\sigma"] \arrow[d, "\pi_A\sigma"'] & B \arrow[d, "-\tau i_B"'] \\ A \arrow[r, "\tau i_A"] & C \end{tikzcd}$$

where the difference of the two paths is $\tau\sigma$.

Characterization of pullbacks

$$\begin{array}{ccc} P & \xrightarrow{\delta} & B \\ {\scriptstyle\gamma}\big\downarrow & & {\scriptstyle\beta}\big\downarrow \\ A & \xrightarrow{\alpha} & C \end{array}$$ is a pullback square iff $0 \longrightarrow P \longrightarrow A \oplus B \longrightarrow C$ is exact.

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Sketch of Proof. $(\Leftarrow)$

From commutative $\begin{tikzcd} X \arrow[r, "\nu"] \arrow[d, "\mu"'] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"] & C, \end{tikzcd}$ get complex $\begin{tikzcd} X \arrow[d] & \\ A \oplus B \arrow[r] & C. \end{tikzcd}$

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$(\Rightarrow)$ Compare complex $P \longrightarrow A \oplus B \longrightarrow C$ with exact sequence $0 \longrightarrow K \xrightarrow{\kappa} A \oplus B \longrightarrow C$. Use that P is a pullback to show that $P \to A \oplus B$ is equivalent to $K \to A \oplus B$.

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Idea of Proof. Given $\alpha\colon A \to C$ and $\beta\colon B \to C$, construct P as the kernel of the induced map $A \oplus B \to C$.

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is a pullback square and $K \xrightarrow{\kappa} P$ is a kernel of δ, then $K \xrightarrow{\kappa} P \xrightarrow{\gamma} A$ is a kernel of α. (In particular, δ monic implies α monic.)

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Proofs given in class.