Normalized Cochains

A cochain f is *normalized* if $f(x_1, \ldots, x_n) = 0$ whenever some $x_j = 1$.

Theorem 1. *The cohomology groups $H^n(G, A)$ for unnormalized cochains are the same as those for normalized cochains.*

This theorem follows from the following two lemmas.

Lemma 2. *Every unnormalized cocycle is cohomologous to a normalized cocyle.*

Lemma 3. *A normalized coboundary is the image of a normalized cochain under the boundary map.*

In turn, these lemmas follow from Lemma 4. To state it, let f denote an n-cochain and define $f_0 := f$, $f_{i+1} = f_i - \partial g_{i+1}$ where

$$g_{i+1}(x_1, \ldots, x_{n-1}) := (-1)^i f_i(x_1, \ldots, x_i, 1, x_{i+1}, \ldots, x_{n-1}). \tag{1}$$

Since $f_i - f_{i+1} = \partial g_{i+1}$ is a coboundary, we have $f_i \sim f_{i+1}$ for all i, hence $f = f_0 \sim f_i$ for all i. Since cohomologous cochains have the same image under ∂, we have $\partial f = \partial f_i$ for all i.

Call a cochain *i-normalized* if $f(x_1, \ldots, x_n) = 0$ whenever some $x_j = 1$ for $j \in \{1, \ldots, i\}$.

Lemma 4. *If ∂f is normalized, then f_i is i-normalized for all i.*

Proof. The proof is by induction on i, with the case $i = 0$ being trivial. So assume that ∂f is normalized and f is i-normalized, and let's show that f_{i+1} is $(i+1)$-normalized.

Claim 5. *f_{i+1} is i-normalized.*

Since $f_{i+1} = f_i - \partial g_{i+1}$ and f_i is i-normalized, it suffices to prove that ∂g_{i+1} is i-normalized.

$$\begin{aligned} \partial g_{i+1}(x_1, \ldots, x_n) &= x_1 \cdot g_{i+1}(x_2, x_3, \ldots, x_n) - g_{i+1}(x_1 x_2, x_3, \ldots, x_n) \\ &\quad + g_{i+1}(x_1, x_2 x_3, \ldots, x_n) - g_{i+1}(x_1, x_2, x_3 x_4, \ldots, x_n) + \cdots \\ &\quad + (-1)^{n-1} g_{i+1}(x_1, x_2, \ldots, x_{n-1} x_n) + (-1)^n g_{i+1}(x_1, x_2, \ldots, x_{n-1}). \end{aligned} \tag{2}$$

It is clear from (1) that g_{i+1} is i-normalized if f_i is. So if we substitute $x_j = 1$ for any $j \in \{1, \ldots, i\}$, then all terms in (2) become zero except two. The remaining two are equal terms with opposite sign, so they cancel. Hence ∂g_{i+1} is indeed i normalized, and the claim is proved.

To complete the proof that f_{i+1} is $(i+1)$-normalized we must show that $f_{i+1}(x_1, \ldots, x_n) = 0$ whenever $x_{i+1} = 0$. The idea to prove this is simple. We expand

$$f_{i+1}(x_1, \ldots, x_i, 1, x_{i+2}, \ldots, x_n) \tag{3}$$

in terms of f_i and use the fact that f_i is i-normalized to show that many of the terms of the expansion are zero. Then we expand

$$\partial f_i(x_1, \ldots, x_i, 1, 1, x_{i+2}, \ldots, x_n) \tag{4}$$

in terms of f_i and do the same thing. The terms we end up with are the same up to sign. Thus (3) equals zero iff (4) equals zero. But $\partial f = \partial f_i$, so we can replace f_i in (4) with f, and use the hypothesis that ∂f is normalized to prove that (4) is indeed zero. This will complete the proof.

So it remains to expand (3) and (4) and examine what kind of cancellation takes place. Expanding (3) using the definition of f_{i+1} we get

$$
\begin{aligned}
f_{i+1}(x_1, \ldots, x_i, 1, x_{i+2}, \ldots, x_n) \quad &= f_i(x_1, \ldots, x_i, 1, x_{i+2}, \ldots, x_n) \\
&-x_1 \cdot g_{i+1}(x_2, x_3, \ldots, x_i, 1, x_{i+2}, \ldots, x_n) \\
&+\textstyle\sum_{j=1}^{i-1}(-1)^{j-1} g_{i+1}(x_1, \ldots, x_j x_{j+1}, \ldots, x_i, 1, x_{i+2}, \ldots, x_n) \\
&(-1)^{i-1} g_{i+1}(x_1, \ldots, x_{i-1}, x_i \cdot 1, x_{i+2}, \ldots, x_n) \\
&(-1)^{i} g_{i+1}(x_1, \ldots, x_{i-1}, x_i, 1 \cdot x_{i+2}, \ldots, x_n) \\
&+\textstyle\sum_{j=i+2}^{n-1}(-1)^{j-1} g_{i+1}(x_1, \ldots, x_i, 1, x_{i+2}, \ldots, x_j x_{j+1}, \ldots, x_n) \\
&+(-1)^n g_{i+1}(x_1, \ldots, x_i, 1, x_{i+2}, \ldots, x_{n-1}).
\end{aligned}
\tag{5}
$$

Since f_i is i-normalized, g_{i+1} is i-normalized. Thus, the second and third lines of (5) are zero. The fourth and fifth lines are equal with opposite sign, so they cancel. Let us use the definition of g_{i+1} to rewrite what remains (the first, sixth and seventh lines) in terms of f_i:

$$
\begin{aligned}
f_{i+1}(x_1, \ldots, x_i, 1, x_{i+2}, \ldots, x_n) \quad &= f_i(x_1, \ldots, x_i, 1, x_{i+2}, \ldots, x_n) \\
&+\textstyle\sum_{j=i+2}^{n-1}(-1)^{i+j-1} f_i(x_1, \ldots, x_i, 1, 1, x_{i+2}, \ldots, x_j x_{j+1}, \ldots, x_n) \\
&+(-1)^{n+i} f_i(x_1, \ldots, x_i, 1, 1, x_{i+2}, \ldots, x_{n-1}).
\end{aligned}
\tag{6}
$$

Expanding (4) we get

$$
\begin{aligned}
\partial f_i(x_1, \ldots, x_i, 1, 1, x_{i+2}, \ldots, x_n) \quad &= x_1 \cdot f_i(x_2, \ldots, x_i, 1, 1, x_{i+2}, \ldots, x_n) \\
&+\textstyle\sum_{j=1}^{i-1}(-1)^{j} f_i(x_1, \ldots, x_j x_{j+1}, \ldots, x_i, 1, 1, x_{i+2}, \ldots, x_n) \\
&(-1)^{i} f_i(x_1, \ldots, x_{i-1}, x_i \cdot 1, 1, x_{i+2}, \ldots, x_n) \\
&(-1)^{i+1} f_i(x_1, \ldots, x_{i-1}, x_i, 1 \cdot 1, x_{i+2}, \ldots, x_n) \\
&(-1)^{i+2} f_i(x_1, \ldots, x_{i-1}, x_i, 1, 1 \cdot x_{i+2}, \ldots, x_n) \\
&+\textstyle\sum_{j=i+2}^{n-1}(-1)^{j-1} f_i(x_1, \ldots, x_i, 1, 1, x_{i+2}, \ldots, x_j x_{j+1}, \ldots, x_n) \\
&+(-1)^n f_i(x_1, \ldots, x_i, 1, 1, x_{i+2}, \ldots, x_{n-1}).
\end{aligned}
\tag{7}
$$

The first and second lines are zero because f_i is i-normalized. The next three lines are equal but with alternating signs, so lines three and four can be canceled. Multiplying by $(-1)^i$ we get

$$
\begin{aligned}
(-1)^i \partial f_i(x_1, \ldots, x_i, 1, 1, x_{i+2}, \ldots, x_n) \quad &= f_i(x_1, \ldots, x_{i-1}, x_i, 1, x_{i+2}, \ldots, x_n) \\
&+\textstyle\sum_{j=i+2}^{n-1}(-1)^{i+j-1} f_i(x_1, \ldots, x_i, 1, 1, x_{i+2}, \ldots, x_j x_{j+1}, \ldots, x_n) \\
&+(-1)^{n+i} f_i(x_1, \ldots, x_i, 1, 1, x_{i+2}, \ldots, x_{n-1}),
\end{aligned}
\tag{8}
$$

proving that (3) and (4) have the same expansions up to sign. Since $\partial f_i = \partial f$ is normalized, the expansion of (4) is zero, hence the expansion of (3) is also zero, completing the proof that f_{i+1} is $(i+1)$-normalized. $\square$