HOMOLOGICAL ALGEBRA
HOMEWORK ASSIGNMENT VI

Read pages 160-189.

PROBLEMS

Chriestenson Keller Pratarelli Gern Selker Lizzi Wakefield Tuley Hower Martinez Moorhead Li Scherer

1. (Tuley, Li) Calculate $H^2(\mathbb{Z}_4, \mathbb{Z}_2)$.

2. (Chriestenson, Pratarelli) Calculate $H^1(\mathrm{Aut}(A), A)$ when A is the Klein group.

3. (Keller, Moorhead) Suppose that $G = \{1, g\}$ has two elements, and that A is a G-module.

(a) Show that for $n > 0$

$$H^n(G, A) = \begin{cases} \ker(g+1)/\operatorname{im}(g-1) & n \text{ odd;} \\ \ker(g-1)/\operatorname{im}(g+1) & n \text{ even.} \end{cases}$$

(b) Let $A = \mathbb{C}^*$ and let $G = \{1, g\}$ where g is complex conjugation. Show that $H^n(G, \mathbb{C}^*) = \{0\}$ for even $n > 0$ and $H^n(G, \mathbb{C}^*) \cong \mathbb{Z}_2$ for odd n.

4. (Hower, Scherer, Wakefield) Let $\mathbb{E} = \mathbb{F}[\sqrt{\alpha}]$ be such that $\mathbb{E}/\mathbb{F}$ is a nontrivial Galois extension. Let $G = \mathrm{Gal}(\mathbb{E}/\mathbb{F})$.

(a) Show that $H^2(G, \mathbb{E}^*)$ is isomorphic to $\mathbb{F}^*/N$ where N is the group of *norm elements* (those elements of the form $N(u + \sqrt{\alpha}v) = u^2 - \alpha v^2$ with $u, v \in \mathbb{F}$).

(b) Show that if $\mathbb{E}/\mathbb{F} = \mathbb{C}/\mathbb{R}$, then $H^2(G, \mathbb{E}^*) \cong \mathbb{Z}_2$.

(c) Show that if $\mathbb{E}/\mathbb{F} = \mathbb{Q}[\sqrt{2}]/\mathbb{Q}$, then $H^2(G, \mathbb{E}^*)$ is infinite.

(Hint for (c): There are infinitely many primes p for which $x^2 \equiv 2 \pmod p$ is not solvable in $\mathbb{Z}$. For such p, show that any normalized cocyle satisfying $f(\alpha, \alpha) \in p\mathbb{Z} - p^2\mathbb{Z}$ is not cohomologous to zero. Hence at least $p - 1$ cocyles are pairwise noncohomologous for each such p.)

5. (Gern, Lizzi) Describe all central extensions of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by $\mathbb{Z}_2$ up to equivalence. Explain why some inequivalent extensions have isomorphic middle factors.

6. (Martinez, Selker) Given a homomorphism $\varphi\colon G' \to G$, one can convert any G-module into a G'-module by restriction of scalars. Show that such a function induces an additive homomorphism on n-cochains:

$$\widehat{\varphi}\colon C^n(G,A) \to C^n(G',A)\colon f \mapsto f \circ \varphi^n.$$

Show that $\widehat{\varphi}$ maps cocyles to cocycles and coboundaries to coboundaries, hence determines a homomorphism $H^n(G,A) \to H^n(G',A)$.