HOMOLOGICAL ALGEBRA
HOMEWORK ASSIGNMENT V

Read pages 73-90.

PROBLEMS

1. (Chriestenson, Moore) Exercise 7.16 of Atiyah-Macdonald.

2. (Keller, Pratarelli) Let A and B be R-modules and let A' be a submodule of A. For a homomorphism $\varphi\colon A' \to B$ define the *obstruction* of φ to be $\delta(\varphi)$ where $\delta\colon \operatorname{Hom}(A', B) \to \operatorname{Ext}^1(A/A', B)$ is the connecting homomorphism. Show that $\varphi\colon A' \to B$ can be extended to a homomorphism from A to B iff its obstruction is 0.

3.(Gern, Selker) Let D be an abelian group. Show that D is divisible iff $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, D) = \{0\}$.

4. (Lizzi, Wakefield) Let A and B be abelian groups.

(a) Show that if $mA = \{0\}$, then $m\operatorname{Ext}^1_{\mathbb{Z}}(A, B) = \{0\}$ and that if $nB = \{0\}$, then $n\operatorname{Ext}^1_{\mathbb{Z}}(A, B) = \{0\}$.

(b) Show that if A is uniquely m-divisible (i.e., the map $a \mapsto ma$ is an isomorphism of A onto A), then $\operatorname{Ext}^1_{\mathbb{Z}}(A, B)$ is uniquely m-divisible, and that if B is uniquely n-divisible, then $\operatorname{Ext}^1_{\mathbb{Z}}(A, B)$ is uniquely n-divisible.

5. (Jones, Tuley) Give an example to show that the Universal Coefficient Theorem for Homology may fail for complexes that are not flat.

A basic result about finitely generated abelian groups is that each one splits into a direct sum of its torsion subgroup and a torsion-free complement. This is equivalent to the condition that $\operatorname{Ext}^1_{\mathbb{Z}}(A/t(A), t(A)) = 0$ whenever A is finitely generated. The next two exercises explore to what degree this splitting result extends to infinitely generated abelian groups.

6. (Hower, Martinez, Moorhead) (If $t(A)$ has bounded exponent, then $0 \to t(A) \to A \to A/t(A) \to 0$ splits.) Show that every extension of an abelian group B of bounded exponent by a torsion-free abelian group F splits. (Hints: (i) F is flat, so tensoring $0 \to \mathbb{Z} \to \mathbb{Q}$ shows that $F \cong Z \otimes F$ is embeddable in the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes F =: V$. (ii) The LES obtained from $0 \to F \to V \to V/F \to 0$ and $\operatorname{Hom}(\_, B)$ yields an exact sequence

$\mathrm{Ext}^1(V, B) \to \mathrm{Ext}^1(F, B) \to 0$. **(iii) Using the result of Exercise 4, deduce that** $\mathrm{Ext}^1(V, B)$ **is divisible and** $\mathrm{Ext}^1(F, B)$ **has bounded exponent. Complete the argument.)**

7. (Li, Scherer) (There exist abelian groups where $0 \to t(A) \to A \to A/t(A) \to 0$ **does not split.)**

(a) Show that $t(\prod_{p\ \mathbf{prime}} \mathbb{Z}_p) = \oplus_{p\ \mathbf{prime}} \mathbb{Z}_p$.

(b) If D **is defined by**

$$(*) \qquad 0 \to \bigoplus_{p\ \mathbf{prime}} \mathbb{Z}_p \to \prod_{p\ \mathbf{prime}} \mathbb{Z}_p \to D \to 0,$$

then show that some extension of $\bigoplus_{p\ \mathbf{prime}} \mathbb{Z}_p$ **by** D **does not split.**

(Hints for (b): (i) Show that D **is torsion-free and divisible, hence a** $\mathbb{Q}$**-vector space. (ii) Using that** $\mathrm{Ext}^n(\mathbb{Q}, \prod_{p\ \mathbf{prime}} \mathbb{Z}_p) \cong \prod_{p\ \mathbf{prime}} \mathrm{Ext}^n(\mathbb{Q}, \mathbb{Z}_p)$ **and the LES associated to** $(*)$ **and** $\mathrm{Hom}(\mathbb{Q}, \_)$**, show that** $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, D) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \bigoplus_{p\ \mathbf{prime}} \mathbb{Z}_p)$ **(iii) Deduce that** $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \bigoplus_{p\ \mathbf{prime}} \mathbb{Z}_p) \neq 0$. **Complete the argument.)**