HOMOLOGICAL ALGEBRA
HOMEWORK ASSIGNMENT IV

Read pages 30-73.

PROBLEMS

1. (Martinez, Moore) Suppose T is a right exact functor and some derived functor LT_n, $n > 0$, is also right exact. Show that LT_m is the zero functor for all $m \geq n$.

2. (Hower, Jones, Selker)

(a) Show that any $\mathbb{Z}_4$-module is isomorphic to one of the form $(\bigoplus^\kappa \mathbb{Z}_2) \oplus \left(\bigoplus^\lambda \mathbb{Z}_4\right)$ for some κ and λ.

(b) Explain how to determine the isomorphism type of $\operatorname{Tor}_n^{\mathbb{Z}_4}(A, B)$ for any $\mathbb{Z}_4$-modules A and B. You may use the fact that Tor distributes over direct sums.

(Hint for (a): If M is a $\mathbb{Z}_4$-module, let $X \subseteq M$ be maximally independent over $\mathbb{Z}_4$. Let $F = \langle X \rangle$ be the (free) submodule generated by X. If $M[2]$ is the submodule of M annihilated by $2 \in \mathbb{Z}_4$, then $2F \leq M[2]$. Choose $C \leq M[2]$ such that $M[2] = C \oplus 2F$. Show that $M = C \oplus F$ and that C is a direct sum of copies of $\mathbb{Z}_2$ while F is a direct sum of copies of $\mathbb{Z}_4$.)

3. (Pratarelli, Scherer) Compute $\operatorname{Tor}_n^{\mathbb{Z}_m}(\mathbb{Z}_a, \mathbb{Z}_b)$ if $a, b \mid m$.

4. (Tuley, Wakefield) Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules.

(a) Show that if C is flat, then A is flat iff B is.

(b) Give an example to show that the bi-implication in (a) may fail when C is not flat.

5. (Lizzi, Moorhead) A SES $0 \to A \to B \to C \to 0$ is *pure exact* if it remains exact under tensoring with an arbitrary module. Show that C is flat iff any SES ending in C is pure exact. (Hint for "only if": Choose B to be free, so that $\operatorname{Tor}_1(X, B) = 0$. This forces the connecting homomorphism $\delta\colon \operatorname{Tor}_1(X, C) \to X \otimes A$ to be monic. Use purity to deduce $\operatorname{Tor}_1(X, C) = 0$.)

6. (Chriestenson, Li) Weibel Exercise 3.1.3.

7. (Gern, Keller) Weibel Exercise 3.2.3.