HOMOLOGICAL ALGEBRA
HOMEWORK ASSIGNMENT III

Read pages 5-29.

Terminology: exact functor = functor preserving exact sequences. Additive functor = a functor $F\colon \mathcal{C} \to \mathcal{D}$ between additive categories such that the induced function $F\colon \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(F(A), F(B))$ is a group homomorphism.

PROBLEMS

1. (Kearnes) Prove the theorem about the exact homology sequence for any abelian category without using any tricks.

2. (Li, Lizzi) Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of complexes whose differentiations are all zero. Explicitly describe the connecting homomorphisms, explicitly determine the exact homology sequence, and verify the exactness of this sequence.

3. (Moore, Scherer)

(a) Prove the Strong Four Lemma: if the following diagram of modules commutes, has exact rows, α is epic, and δ is monic, then $\ker(\gamma) = \sigma(\ker(\beta))$ and $\mathrm{im}(\beta) = \tau^{-1}(\mathrm{im}(\gamma))$.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r, "\sigma"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[d, "\delta"'] \\
A' \arrow[r] & B' \arrow[r, "\tau"] & C' \arrow[r] & D'
\end{tikzcd}$$

(b) Use the Strong Four Lemma to prove the Five Lemma: if the rows of the following diagram are exact, and

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r] \arrow[d, "\delta"'] & E \arrow[d, "\varepsilon"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

(i) if α is epic, β and δ are monic, then γ is monic.
(ii) if ε is monic, β and δ are epic, then γ is epic.
(iii) If $\alpha, \beta, \delta, \varepsilon$ are all isomorphisms, then so is γ.

4. (Keller, Martinez) Prove that the composition of two nullhomotopic chain maps is again nullhomotopic.

5. (Gern, Hower) Do Exercise 1.4.3 of Weibel.

6. (Moorhead, Tuley) Prove that homology commutes with exact additive functors. That is, if $F\colon {}_R\mathrm{Mod} \to {}_S\mathrm{Mod}$ is an exact additive functor, then for every complex C of R-modules and every $n \in \mathbb{Z}$ there is an S-module isomorphism $H_n(F(C)) \cong F(H_n(C))$.

7. (Chriestenson, Jones) Prove that homology commutes with direct sums: $H_n(\oplus C_i) \cong \oplus H_n(C_i)$.

8. (Pratarelli, Selker) The *torsion product* of abelian groups A and B, written $\mathrm{Tor}(A, B)$, is presented by generators and relations with the generators being all triples $(a, m, b) \in A \times \mathbb{Z} \times B$ satisfying $ma = 0$ and $mb = 0$, and the relations all those of the form

- $(a_1 + a_2, m, b) = (a_1, m, b) + (a_2, m, b)$, **provided** $ma_i = 0 = mb$,
- $(a, m, b_1 + b_2) = (a, m, b_1) + (a, m, b_2)$, **provided** $ma = 0 = mb_i$,
- $(a, mn, b) = (ma, n, b)$, **provided** $mna = 0 = nb$, **and**
- $(a, mn, b) = (a, m, nb)$, **provided** $ma = 0 = mnb$.

Show that if A and B are finite, then $\mathrm{Tor}(A, B) \cong A \otimes B$, but that $\mathrm{Tor}(\mathbb{Z}, \mathbb{Z}) \not\cong \mathbb{Z} \otimes \mathbb{Z}$. (Hint for the first part: try proving it for cyclic groups first.)