HOMOLOGICAL ALGEBRA
HOMEWORK ASSIGNMENT II

Read pages 424-431. Terminology: Ab-category = preadditive category.

PROBLEMS

1. (Selker, Wakefield)

(a) Show that $\mathbb{Z}_2$ is a cogroup in the category of abelian groups.
(b) Show that $\mathbb{Z}_2$ is not a cogroup in the category of all groups.
(c) Is $\mathbb{Z}_2$ is a cogroup in the category of groups of exponent 4?

2. (Moorhead, Scherer) Determine which of the following categories are abelian.

(a) The category of torsion abelian groups.
(b) The category of torsion-free abelian groups.
(c) The category of finitely generated abelian groups.
(d) The category of divisible abelian groups.

3. (Chriestenson, Hower) Imagine the integer polynomial 'ring' in a proper class of noncommuting variables:

$$R = \mathbb{Z}\langle\{x_\alpha \mid \alpha \text{ an ordinal}\}\rangle.$$

Let $\mathcal{C}$ be the category of "R-modules", by which we mean the category whose objects are ordinary abelian groups equipped with one endomorphism for each x_α. The morphisms of $\mathcal{C}$ are the abelian group homomorphisms that respect the x_α's.

(a) Show that $\mathcal{C}$ is an abelian category.
(b) Show that $\mathcal{C}$ is not equivalent to a category of modules over a true ring.

4. (Gern, Jones) Let $\mathcal{C}$ be an abelian category. Show that if μ is monic and κ is epic, then μ is a kernel of κ iff κ is a cokernel of μ.

5. (Moore, Pratarelli, Tuley) Prove the following strong form of the remark made at the bottom of page 425.

(a) **If $\mathcal{C}$ is an abelian category and $A \xrightarrow{\varphi} B$ is a morphism in $\mathcal{C}$, then φ can be factored as $A \xrightarrow{\varepsilon} I \xrightarrow{\mu} B$ where $\varepsilon = \operatorname{coker}(\ker(\varphi))$ and $\mu = \ker(\operatorname{coker}(\varphi))$. Show that ε is epi and μ is monic.**

(b) **If**

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A' & \xrightarrow{\varphi'} & B' \end{array}$$

is a commutative square and φ' also has an epi-mono factorization $A' \xrightarrow{\varepsilon'} I' \xrightarrow{\mu'} B'$, then there is a unique morphism $I \xrightarrow{\gamma} I'$ such that the following commutes:

$$\begin{array}{ccccc} A & \xrightarrow{\varepsilon} & I & \xrightarrow{\mu} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} \\ A' & \xrightarrow{\varepsilon'} & I' & \xrightarrow{\mu'} & B'. \end{array}$$

6. (Li, Martinez)

(a) **Prove that in any category with a zero object the projections of a product are epimorphisms and the coprojections of a coproduct are monomorphisms.**

(b) **Prove that in an abelian category a morphism is an isomorphism iff it is both a monomorphism and an epimorphism.**

7. (Keller, Lizzi) Let $\mathcal{A}$ be an abelian category. Given an object $A \in \mathcal{A}$, define an equivalence relation on pairs (X, α), where $\alpha\colon X \to A$ is a morphism, by $(X, \alpha) \sim (X', \alpha')$ if there exist $Y \in \mathcal{A}$ and epimorphisms β, β' such that

$$\begin{array}{ccc} Y & \xrightarrow{\beta} & X \\ {\scriptstyle\beta'}\downarrow & & \downarrow{\scriptstyle\alpha} \\ X' & \xrightarrow{\alpha'} & A \end{array}$$

commutes. By a *point* of A we mean an equivalence class $a = [(X, \alpha)]$. (Write $a \in A$.)

(a) **Show that $\sim$ is an equivalence relation.**

If $\varphi\colon A \to B$ and $a = [(X, \alpha)] \in A$, then let $\varphi(a) := [(X, \varphi \circ \alpha)]$ $(\in B)$.

(b) **Show that if $\varphi, \psi\colon A \to B$ and $\varphi(a) = \psi(a)$ for all points of A, then $\varphi = \psi$.**