

Categories and Functors

The intuitive model of a category is a collection of mathematical structures (called *objects*) equipped with structure preserving mappings (called *morphisms*). The formal definition closely resembles our definition of “algebra”; in fact, a (small) category is a 2-sorted partial algebra.

Definition 1. A *category* is a structure

$$\mathcal{C} = \langle O, M; \circ, \text{id}, \text{dom}, \text{cod} \rangle$$

where

- (1) $\text{Ob}(\mathcal{C}) = O$ is a class whose members are called *objects*,
- (2) $\text{Mor}(\mathcal{C}) = M$ is a class whose members are called *morphisms*,
- (3) $\circ : M \times M \rightarrow M$ is a binary partial operation called *composition*,
- (4) $\text{id} : O \rightarrow M$ is a unary function assigning to each object $A \in O$ a morphism id_A called *the identity of A*,
- (5) $\text{dom}, \text{cod} : M \rightarrow O$ are unary functions assigning to each morphism f objects called the *domain* and *codomain* of f respectively.

The laws defining categories are:

- (1) $f \circ g$ exists if and only if $\text{dom}(f) = \text{cod}(g)$.
- (2) Composition is associative when it is defined.
- (3) $\text{dom}(f \circ g) = \text{dom}(g)$, $\text{cod}(f \circ g) = \text{cod}(f)$.
- (4) If $A = \text{dom}(f)$ and $B = \text{cod}(f)$, then $f \circ \text{id}_A = f$ and $\text{id}_B \circ f = f$.
- (5) $\text{dom}(\text{id}_A) = \text{cod}(\text{id}_A) = A$.

We will say that a category is *small* if M is a set. (This forces O to be a set, too.)

Notation. We let $\text{Hom}_{\mathcal{C}}(A, B)$ denote the class of $f \in M$ for which $\text{dom}(f) = A$ and $\text{cod}(f) = B$. It is common to add to the definition of a category the assumption that $\text{Hom}_{\mathcal{C}}(A, B)$ is a set for all $A, B \in O$. (I will always assume this.)

Examples.

- (1) The category of all topological spaces equipped with continuous maps is a category, **Top**. The category of all topological spaces equipped with homotopy classes of continuous maps is a category, **Toph**.
- (2) The category of all groups equipped with group homomorphisms is a category, **Grp**.
- (3) If $\langle P; \leq \rangle$ is a partially ordered set, then the elements of P may be thought of as the objects of a category whose morphisms are the arrows $a \rightarrow b$ where $a \leq b$ in P .
- (4) If $\langle M; \circ, 1 \rangle$ is a monoid, then M determines a one-object category $\mathcal{M} = \langle \{*\}, M; \circ, 1, \text{dom}, \text{cod} \rangle$ where $\text{dom}, \text{cod} : M \rightarrow \{*\}$ are both the constant function.

Terminology 2. *Monomorphism, epimorphism, isomorphism, retraction, section, initial object, terminal object, opposite category.*

Since the definition of a category is so close to that of an algebra, it is natural to try to compare categories with “homomorphisms”. These are called “covariant functors”.

Definition 3. A *covariant functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ is a “structure preserving mapping” from \mathcal{C} to \mathcal{D} . Precisely, F is a pair of mappings, both called F , between object classes and morphism classes, $F(\text{Ob}(\mathcal{C})) \subseteq \text{Ob}(\mathcal{D})$ and $F(\text{Mor}(\mathcal{C})) \subseteq \text{Mor}(\mathcal{D})$, where

- (1) $F(f \circ g) = F(f) \circ F(g)$,
- (2) $F(\text{id}_A) = \text{id}_{F(A)}$,
- (3) $F(\text{dom}(f)) = \text{dom}(F(f))$, and
- (4) $F(\text{cod}(f)) = \text{cod}(F(f))$.

A *contravariant functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ is a “composition reversing mapping” from \mathcal{C} to \mathcal{D} . That is, $F(\text{Ob}(\mathcal{C})) \subseteq \text{Ob}(\mathcal{D})$, $F(\text{Mor}(\mathcal{C})) \subseteq \text{Mor}(\mathcal{D})$, and

- (1) $F(f \circ g) = F(g) \circ F(f)$,
- (2) $F(\text{id}_A) = \text{id}_{F(A)}$,
- (3) $F(\text{dom}(f)) = \text{cod}(F(f))$, and
- (4) $F(\text{cod}(f)) = \text{dom}(F(f))$.

Examples.

- (I) Covariant:
 - (a) Constant functors.
 - (b) Monotone maps between posets.
 - (c) Monoid homomorphisms.
 - (d) Covariant power set functor.
 - (e) Forgetful functor.
 - (f) Free object functor.
 - (g) Unit group functor. (From rings to groups.)
 - (h) GL_n . (From rings to groups.)
- (II) Contravariant:
 - (a) Constant functors.
 - (b) Antitone maps between posets.
 - (c) Monoid antihomomorphisms.
 - (d) Contravariant power set functor.
 - (e) Ring of continuous real-valued functions.
 - (f) Dual space functor.

Terminology 4. *Full, faithful, representative.*