Definitions of (Co)homology in Algebra: Groups and Associative Algebras

Main topics: modules, enveloping algebras, semidirect products, (co)invariants, (co)homology.

I. Groups

Definition 1. Let G be a group. A *G-module* is an abelian group U equipped with a homomorphism $\varphi\colon G \to \mathrm{Aut}(U)$. (We often write gu or $g \cdot u$ for $\varphi(g)(u)$.)

Intuition/theorem: U is a G-module iff there is an exact sequence of groups $0 \to U \xrightarrow{i} A \xrightarrow{\pi} G \to 0$ with U abelian such that for $g \in G$ and $\hat{g} \in \pi^{-1}(g)$ it is the case that $\gamma_{\hat{g}}(u) = gu$.

Definition 2. The *enveloping ring* of G is $\mathbb{Z}[G]$.

There is an obvious adjunction $\mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[G], R) \cong \mathrm{Hom}_{\mathrm{Grp}}(G, R^*)$ converting between G-modules and $\mathbb{Z}[G]$-modules.

If U is a G-module, then an element $u \in U$ is *G-invariant* (or a *fixed point*) if $gu = u$ for all $g \in G$. The collection U^G of all G-invariant elements is a submodule of U, and the map $U \mapsto U^G$ is the object part of a functor. The morphism part is just restriction: if $\alpha\colon U \to V$, then $\alpha|_{U^G}\colon U^G \to V^G$ is the associated morphism. This functor is left exact but not fully exact.

Example 3. ($U \mapsto U^G$ is not left exact) Let the 2-element group G act on abelian groups by inversion. The usual exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ of abelian groups is an exact sequence of G-modules. The sequence of invariant modules $0 \to 0 \to 0 \to \langle \left[\frac{1}{2}\right] \rangle \to 0$ is not exact on the right.

To see that this not-fully-exact functor is left exact we identify it with a hom functor. Observe that $u \in U$ is invariant iff it is annihilated by $g - 1$; i.e., $gu = u$ iff $(g-1)u = 0$. Each element of U is the image of a unique $\mathbb{Z}[G]$-module homomomorphism $\mathbb{Z}[G] \to U$, namely the one ($= \rho_u$) that sends 1 to u. The element $u \in U$ is invariant iff $g - 1 \in \ker(\rho_u)$ for all $g \in G$, equivalently iff ρ_u factors through the *augmentation map* $\mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z}\colon G \mapsto 1$. One can check that there is a natural isomorphism from the invariants functor $U \mapsto U^G$ and the functor $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, U)$.

The invariants submodule U^G is the maximal G-submodule that consists of invariant elements. Similarly we can define the module U_G of coinvariants of U to be the maximal G-quotient module that consists of invariants, $U/\langle\{(g-1)u\}\rangle$. The map $U \mapsto U_G$ is the object part of a functor that is naturally isomorphic to $\mathbb{Z} \otimes_{\mathbb{Z}[G]} \_\_$.

Definition 4.

- $H^n(G, U) := \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, U)$.
- $H_n(G, U) := \mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, U)$.

II. Associative algebras

Definition 5. Let A be an associate algebra over a commutative ring k.
(Version 1) An (A,A)-*bimodule* is a k-module U equipped with a left A-action $A\times U\to U\colon (a,u)\mapsto au$ and a right A-action $U\times A\to U\colon (u,a)\mapsto ua$ such that the associative law holds: $(a_1u)a_2 = a_1(ua_2)$.

(Version 2) An (A,A)-*bimodule* is a k-module U equipped with a pair of homomorphisms
- $\lambda\colon A\to \mathrm{End}_k(U)$
- $\rho\colon A^{op}\to \mathrm{End}_k(U)$

such that the images of λ and ρ commute.

Intuition/theorem: U is an (A,A)-bimodule iff there is an exact sequence of k-algebras $0\to U\xrightarrow{i} B\xrightarrow{\pi} A\to 0$ with U abelian such that for $a\in A$ and $\hat{a}\in\pi^{-1}(a)$ it is the case that $T^{(1,1)}_{(a,*)}(u)=\lambda_a(u)=\hat{a}u$ and $T^{(1,2)}_{(*,a)}(u)=\rho_a(u)=u\hat{a}$.

Definition 6. The *enveloping algebra* of A is $A\otimes A^{op}$.

There is an adjunction $\mathrm{Hom}_{\mathrm{Ring}}(A\otimes_k A^{op},R)\cong \mathrm{Hom}_{\mathcal{D}}(A,R)$ where objects in $\mathcal{D}$ are k-algebras and maps are pairs (λ,ρ) where $\lambda\colon A\to R$ is an algebra homomorphism, $\rho\colon A\to R$ is an anti-homomorphism, and the images of λ and ρ commute.

If U is an (A,A)-bimodule, then an element $u\in U$ is *invariant* if $\lambda_a(u)=\rho_a(u)$ for all $a\in A$. The collection U^A of all A-invariant elements is a submodule of U, and the map $U\mapsto U^A$ is the object part of a left exact functor. An element $u\in U$ is invariant iff the map of bimodules $A\otimes_k A^{op}\to U\colon 1\otimes 1\mapsto u$ factors through the *augmentation map* $A\otimes_k A^{op}\xrightarrow{\varepsilon} A\colon 1\otimes 1\mapsto 1$. One can check that there is a natural isomorphism from the invariants functor $U\mapsto U^G$ and the functor $\mathrm{Hom}_{\mathbb{A}\otimes A^{op}}(A,U)$. Similarly the module of coinvariants can be defined, and the coinvariants functor can be shown (or defined) to be $A\otimes_k \_\_$.

Definition 7.

- $H^n(A,U) := \mathrm{Ext}^n_{A\otimes_k A^{op}}(A,U)$.
- $H_n(A,U) := \mathrm{Tor}_n^{A\otimes_k A^{op}}(A,U)$.

Bimodule actions come up in the definition of semidirect products of associative algebras. If $0\to I\to B\to A\to 0$ is split, then we can identify B with $I\oplus A$ as k-modules and then the multiplication of B must be $(i+a)(i'+a') = (ii'+ia'+i'a)+aa'$. Thus the ring multiplication on B is given by the ring multiplications on I and A together with an (A,A)-binodule action on I. The bimodule action on I should be consistent with the multiplication on I in the sense that $a(ii')=(ai)i'$, $(ii')a=i(i'a)$ and $(ia)i'=i(ai')$.

Definition 8. Given associative k-algebras I and A and a *multiplicative* (A,A)-*bimodule structure* on I the semidirect product $I\rtimes A$ is the k-module $I\oplus A$ equipped with the multiplication $(i,a)(i',a')=(ii'+\rho_{a'}(i)+\lambda_a(i'),aa')$.

Semidirect products describe the structure of split extensions.