Adjoints and Limits

Homological Algebra

March 17, 2010

Adjoints on limits

Thm. Right adjoints preserve limits; left adjoints preserve colimits.

Adjoints on limits

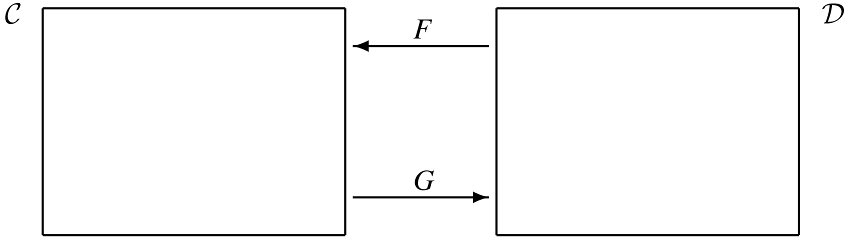
Thm. Right adjoints preserve limits; left adjoints preserve colimits.

Proof. Assume $\varphi\colon \mathrm{Hom}_{\mathcal{C}}(F(Y), X) \to \mathrm{Hom}_{\mathcal{D}}(Y, G(X))$ is a natural isomorphism and $D\colon J \to \mathcal{C}$ is a diagram with $\varprojlim D = (L, \lambda)$.

Adjoints on limits

Thm. Right adjoints preserve limits; left adjoints preserve colimits.

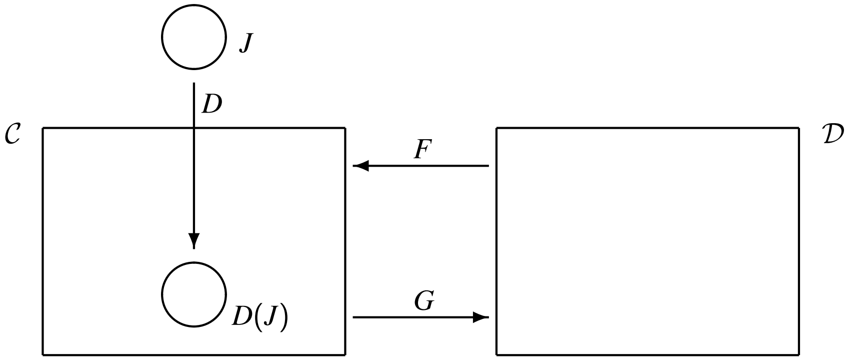
Proof. Assume $\varphi\colon \mathrm{Hom}_{\mathcal{C}}(F(Y), X) \to \mathrm{Hom}_{\mathcal{D}}(Y, G(X))$ is a natural isomorphism and $D\colon J \to \mathcal{C}$ is a diagram with $\varprojlim D = (L, \lambda)$.



Adjoints on limits

Thm. Right adjoints preserve limits; left adjoints preserve colimits.

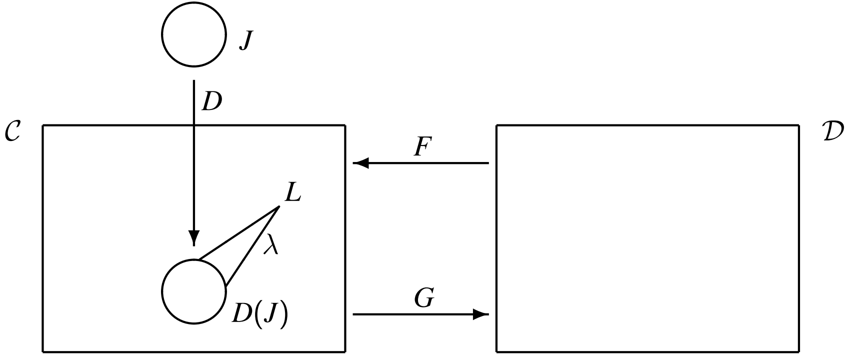
Proof. Assume $\varphi\colon \mathrm{Hom}_{\mathcal{C}}(F(Y), X) \to \mathrm{Hom}_{\mathcal{D}}(Y, G(X))$ is a natural isomorphism and $D\colon J \to \mathcal{C}$ is a diagram with $\varprojlim D = (L, \lambda)$.



Adjoints on limits

Thm. Right adjoints preserve limits; left adjoints preserve colimits.

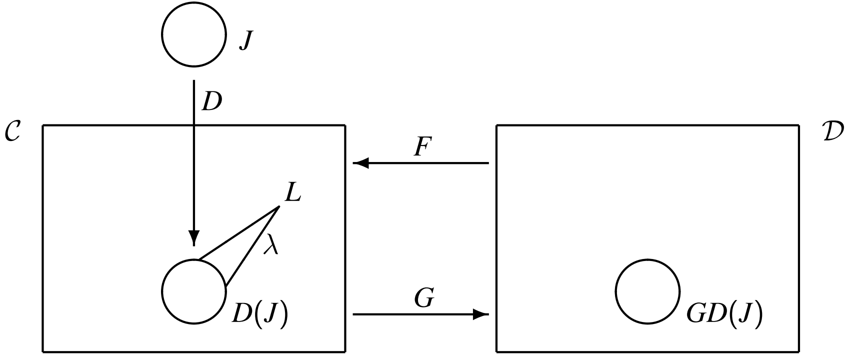
Proof. Assume $\varphi\colon \mathrm{Hom}_{\mathcal{C}}(F(Y), X) \to \mathrm{Hom}_{\mathcal{D}}(Y, G(X))$ is a natural isomorphism and $D\colon J \to \mathcal{C}$ is a diagram with $\varprojlim D = (L, \lambda)$.



Adjoints on limits

Thm. Right adjoints preserve limits; left adjoints preserve colimits.

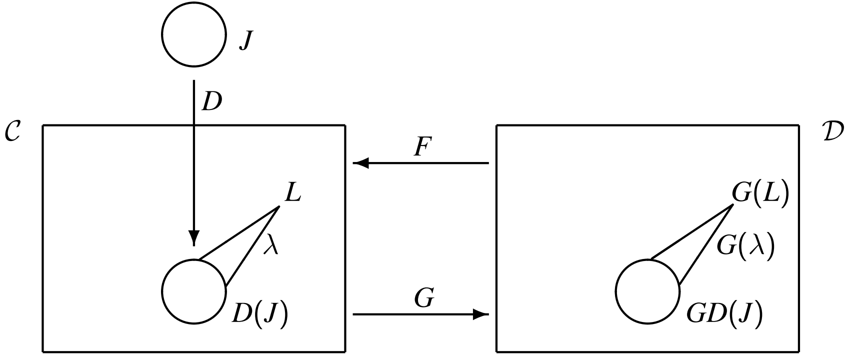
Proof. Assume $\varphi\colon \mathrm{Hom}_{\mathcal{C}}(F(Y), X) \to \mathrm{Hom}_{\mathcal{D}}(Y, G(X))$ is a natural isomorphism and $D\colon J \to \mathcal{C}$ is a diagram with $\varprojlim D = (L, \lambda)$.



Adjoints on limits

Thm. Right adjoints preserve limits; left adjoints preserve colimits.

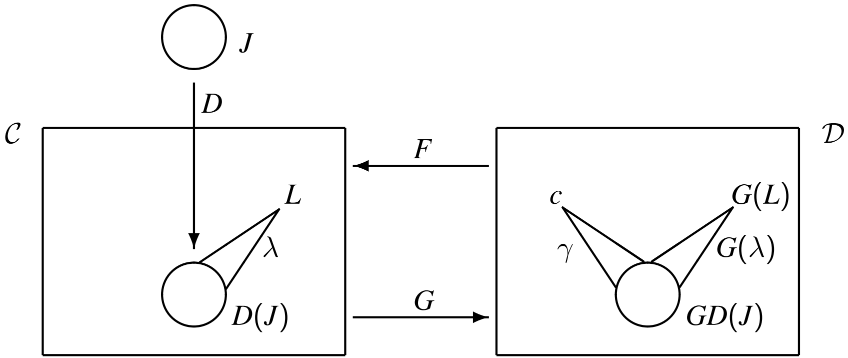
Proof. Assume $\varphi\colon \mathrm{Hom}_{\mathcal{C}}(F(Y), X) \to \mathrm{Hom}_{\mathcal{D}}(Y, G(X))$ is a natural isomorphism and $D\colon J \to \mathcal{C}$ is a diagram with $\varprojlim D = (L, \lambda)$.



Adjoints on limits

Thm. Right adjoints preserve limits; left adjoints preserve colimits.

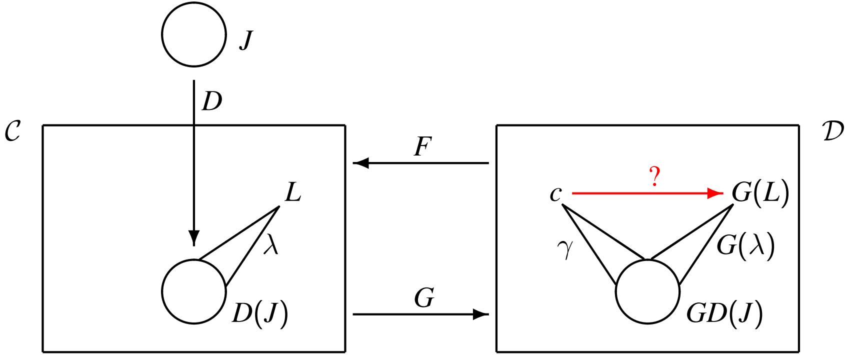
Proof. Assume $\varphi\colon \mathrm{Hom}_{\mathcal{C}}(F(Y), X) \to \mathrm{Hom}_{\mathcal{D}}(Y, G(X))$ is a natural isomorphism and $D\colon J \to \mathcal{C}$ is a diagram with $\varprojlim D = (L, \lambda)$.



Adjoints on limits

Thm. Right adjoints preserve limits; left adjoints preserve colimits.

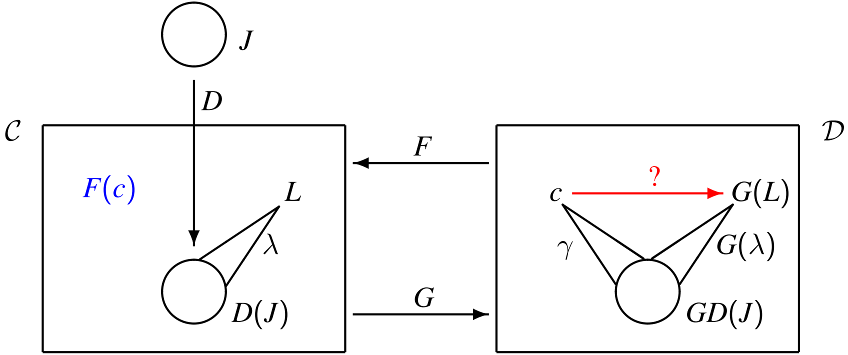
Proof. Assume $\varphi\colon \mathrm{Hom}_{\mathcal{C}}(F(Y), X) \to \mathrm{Hom}_{\mathcal{D}}(Y, G(X))$ is a natural isomorphism and $D\colon J \to \mathcal{C}$ is a diagram with $\varprojlim D = (L, \lambda)$.



Adjoints on limits

Thm. Right adjoints preserve limits; left adjoints preserve colimits.

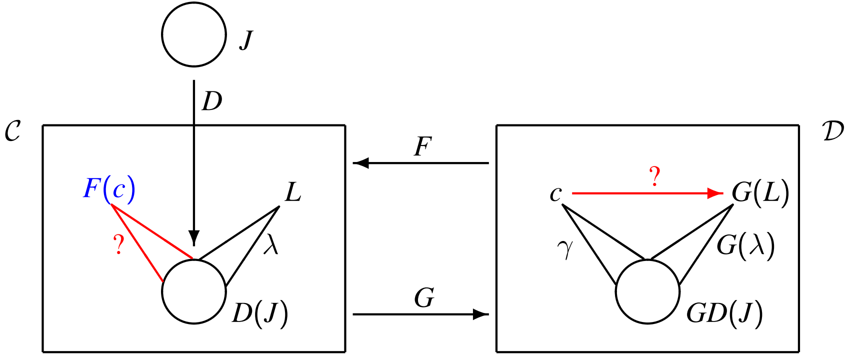
Proof. Assume $\varphi\colon \mathrm{Hom}_{\mathcal{C}}(F(Y), X) \to \mathrm{Hom}_{\mathcal{D}}(Y, G(X))$ is a natural isomorphism and $D\colon J \to \mathcal{C}$ is a diagram with $\varprojlim D = (L, \lambda)$.



Adjoints on limits

Thm. Right adjoints preserve limits; left adjoints preserve colimits.

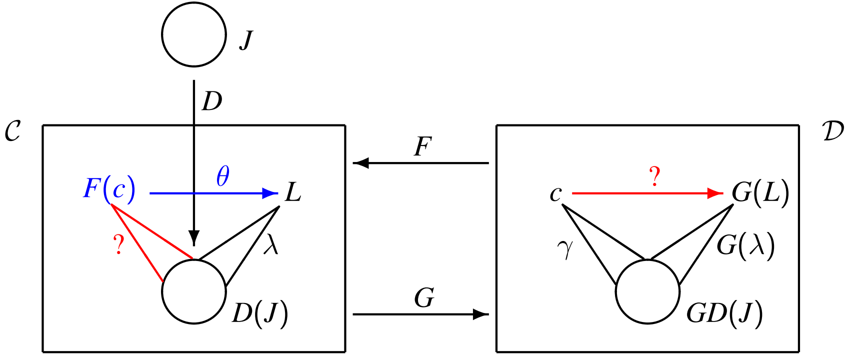
Proof. Assume $\varphi\colon \mathrm{Hom}_{\mathcal{C}}(F(Y), X) \to \mathrm{Hom}_{\mathcal{D}}(Y, G(X))$ is a natural isomorphism and $D\colon J \to \mathcal{C}$ is a diagram with $\varprojlim D = (L, \lambda)$.



Adjoints on limits

Thm. Right adjoints preserve limits; left adjoints preserve colimits.

Proof. Assume $\varphi\colon \mathrm{Hom}_{\mathcal{C}}(F(Y), X) \to \mathrm{Hom}_{\mathcal{D}}(Y, G(X))$ is a natural isomorphism and $D\colon J \to \mathcal{C}$ is a diagram with $\varprojlim D = (L, \lambda)$.



Proof continued

We choose maps from $F(c)$ to $D(J)$ to be those in $\varphi^{-1}(\gamma)$

Proof continued

We choose maps from $F(c)$ to $D(J)$ to be those in $\varphi^{-1}(\gamma)$

To show:

Proof continued

We choose maps from $F(c)$ to $D(J)$ to be those in $\varphi^{-1}(\gamma)$

To show:
(1) $(F(c), \varphi^{-1}(\gamma))$ is a cone to D.

Proof continued

We choose maps from $F(c)$ to $D(J)$ to be those in $\varphi^{-1}(\gamma)$

To show:
(1) $(F(c), \varphi^{-1}(\gamma))$ is a cone to D.
(2) $\varphi(\theta)$ is a cone morphism.

Proof continued

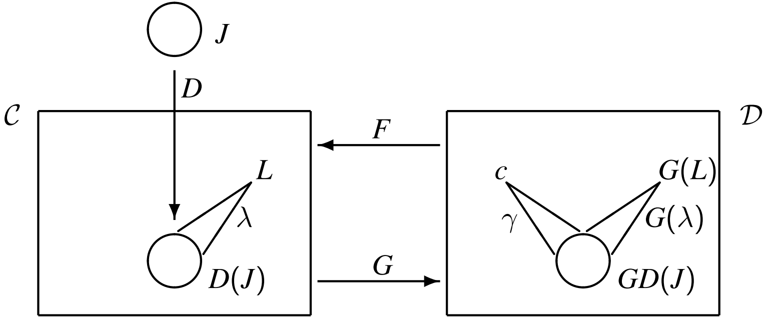
We choose maps from $F(c)$ to $D(J)$ to be those in $\varphi^{-1}(\gamma)$

To show:
(1) $(F(c), \varphi^{-1}(\gamma))$ is a cone to D.
(2) $\varphi(\theta)$ is a cone morphism.
(3) $\varphi(\theta)$ is the unique cone morphism from (c, γ) to $(G(L), G(\lambda))$.

Proof continued

We choose maps from $F(c)$ to $D(J)$ to be those in $\varphi^{-1}(\gamma)$

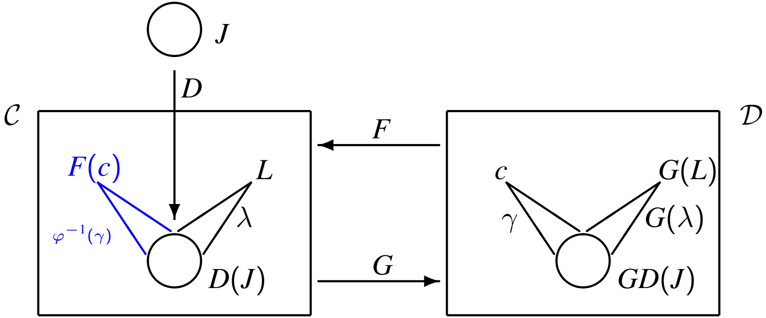
To show:
(1) $(F(c), \varphi^{-1}(\gamma))$ is a cone to D.
(2) $\varphi(\theta)$ is a cone morphism.
(3) $\varphi(\theta)$ is the unique cone morphism from (c, γ) to $(G(L), G(\lambda))$.



Proof continued

We choose maps from $F(c)$ to $D(J)$ to be those in $\varphi^{-1}(\gamma)$

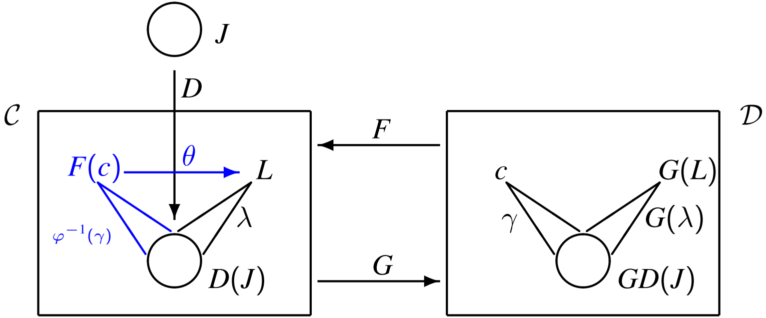
To show:
(1) $(F(c), \varphi^{-1}(\gamma))$ is a cone to D.
(2) $\varphi(\theta)$ is a cone morphism.
(3) $\varphi(\theta)$ is the unique cone morphism from (c, γ) to $(G(L), G(\lambda))$.



Proof continued

We choose maps from $F(c)$ to $D(J)$ to be those in $\varphi^{-1}(\gamma)$

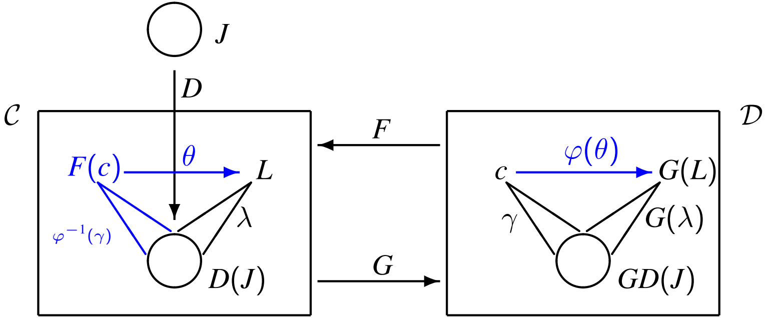
To show:
(1) $(F(c), \varphi^{-1}(\gamma))$ is a cone to D.
(2) $\varphi(\theta)$ is a cone morphism.
(3) $\varphi(\theta)$ is the unique cone morphism from (c, γ) to $(G(L), G(\lambda))$.



Proof continued

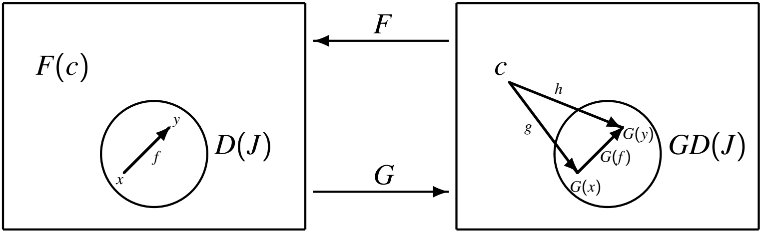
We choose maps from $F(c)$ to $D(J)$ to be those in $\varphi^{-1}(\gamma)$

To show:
(1) $(F(c), \varphi^{-1}(\gamma))$ is a cone to D.
(2) $\varphi(\theta)$ is a cone morphism.
(3) $\varphi(\theta)$ is the unique cone morphism from (c, γ) to $(G(L), G(\lambda))$.

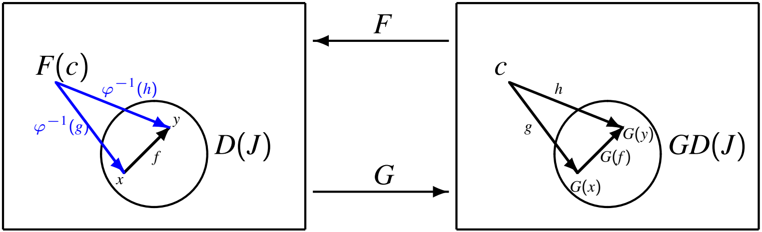


(1) $(F(c), \varphi^{-1}(\gamma))$ is a cone to D

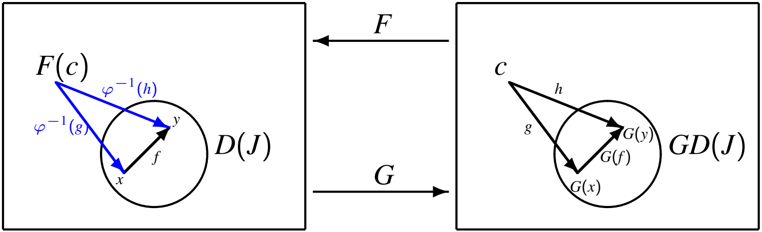
(1) $(F(c), \varphi^{-1}(\gamma))$ is a cone to D



(1) $(F(c), \varphi^{-1}(\gamma))$ is a cone to D

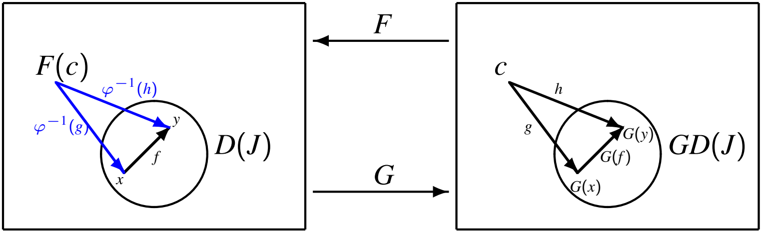


(1) $(F(c), \varphi^{-1}(\gamma))$ is a cone to D



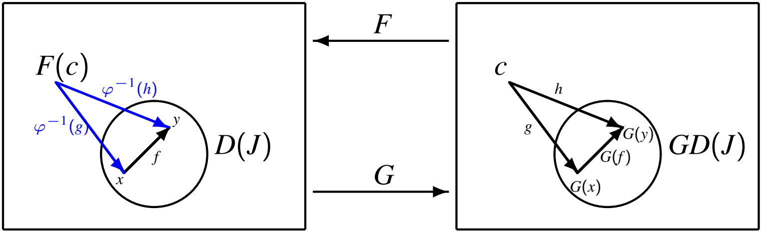
Must show: $h = G(f) \circ g$ implies $\varphi_{c,y}^{-1}(h) = f \circ \varphi_{c,x}^{-1}(g)$.

(1) $(F(c), \varphi^{-1}(\gamma))$ is a cone to D



Must show: $h = G(f) \circ g$ implies $\varphi^{-1}_{c,y}(h) = f \circ \varphi^{-1}_{c,x}(g)$.
Recall: $\varphi^{-1}_{c,y}(h) = \varepsilon_y \circ F(h)$ where $\varepsilon\colon FG \to I$ is the counit of the adjunction.

(1) $(F(c), \varphi^{-1}(\gamma))$ is a cone to D

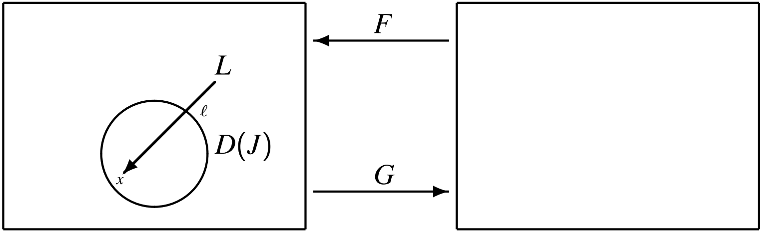


Must show: $h = G(f) \circ g$ implies $\varphi^{-1}_{c,y}(h) = f \circ \varphi^{-1}_{c,x}(g)$.
Recall: $\varphi^{-1}_{c,y}(h) = \varepsilon_y \circ F(h)$ where $\varepsilon \colon FG \to I$ is the counit of the adjunction.

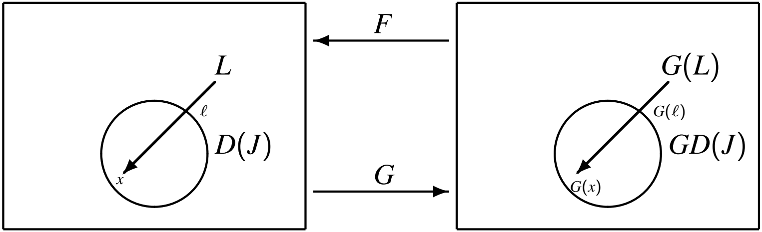
$$\begin{aligned}
\varphi^{-1}_{c,y}(h) &= \varepsilon_y \circ F(h) \\
&= \varepsilon_y \circ F(G(f) \circ g) \\
&= \varepsilon_y \circ F(G(f)) \circ F(g) \\
&= f \circ \varepsilon_x \circ F(g) \\
&= f \circ \varphi^{-1}_{c,x}(g)
\end{aligned}$$

(2) $\varphi(\theta)$ is a cone morphism

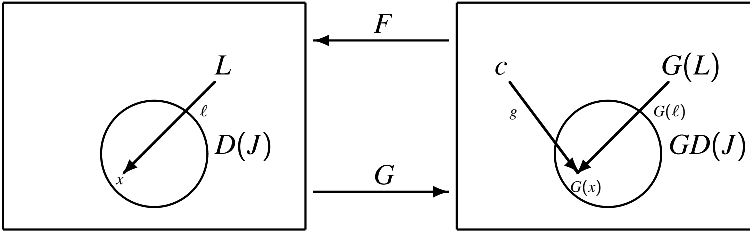
(2) $\varphi(\theta)$ is a cone morphism



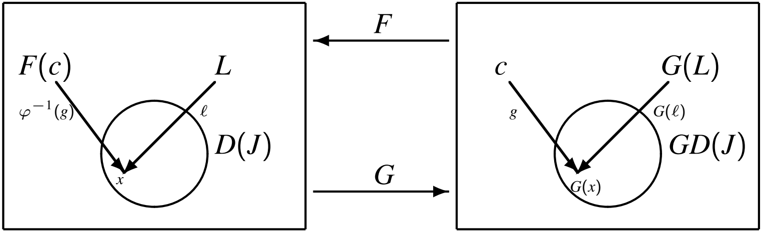
(2) $\varphi(\theta)$ is a cone morphism



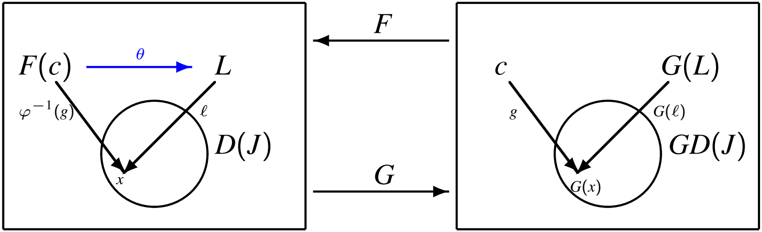
(2) $\varphi(\theta)$ is a cone morphism



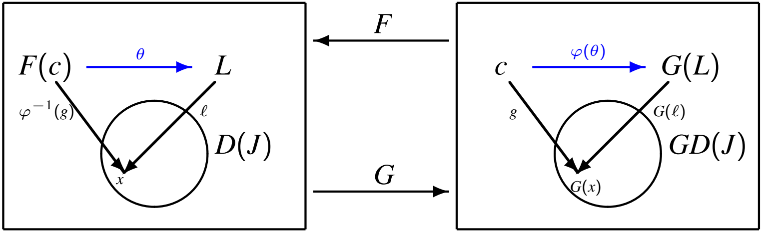
(2) $\varphi(\theta)$ is a cone morphism



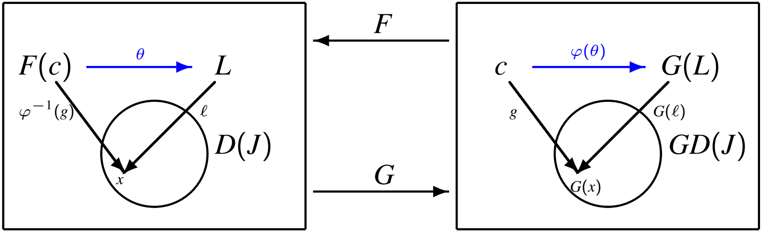
(2) $\varphi(\theta)$ is a cone morphism



(2) $\varphi(\theta)$ is a cone morphism

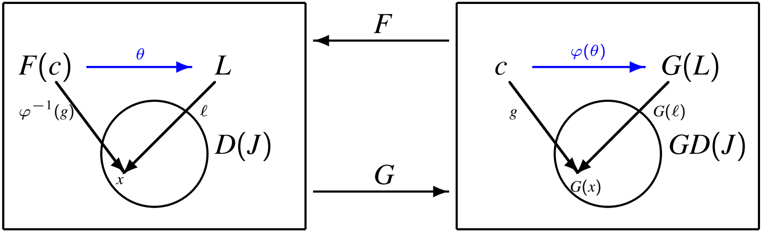


(2) $\varphi(\theta)$ is a cone morphism



Must show: $\ell \circ \theta = \varphi_{c,x}^{-1}(g)$ implies $G(\ell) \circ \varphi_{c,L}(\theta) = g$.

(2) $\varphi(\theta)$ is a cone morphism



Must show: $\ell \circ \theta = \varphi_{c,x}^{-1}(g)$ implies $G(\ell) \circ \varphi_{c,L}(\theta) = g$.

$$\begin{aligned} G(\ell) \circ \varphi_{c,L}(\theta) &= G(\ell) \circ G(\theta) \circ \eta_c \\ &= G(\ell \circ \theta) \circ \eta_c \\ &= G(\varphi_{c,x}^{-1}(g)) \circ \eta_c \\ &= \varphi_{c,x}(\varphi_{c,x}^{-1}(g)) \\ &= g \end{aligned}$$

(3) $\varphi(\theta)$ is unique

(3) $\varphi(\theta)$ is unique

The previous argument showed that if $\theta\colon F(c) \to L$ is a cone map, then $\varphi(\theta)\colon c \to G(L)$ is a cone map.

(3) $\varphi(\theta)$ is unique

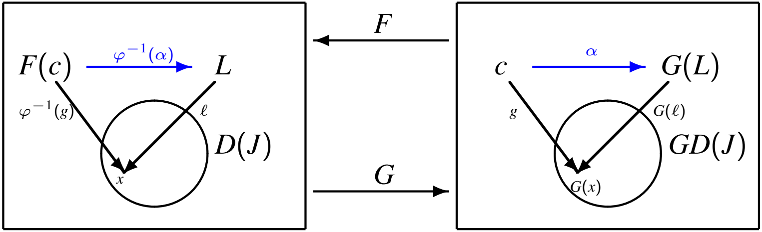
The previous argument showed that if $\theta\colon F(c) \to L$ is a cone map, then $\varphi(\theta)\colon c \to G(L)$ is a cone map.

The reverse is true, too.

(3) $\varphi(\theta)$ is unique

The previous argument showed that if $\theta\colon F(c) \to L$ is a cone map, then $\varphi(\theta)\colon c \to G(L)$ is a cone map.

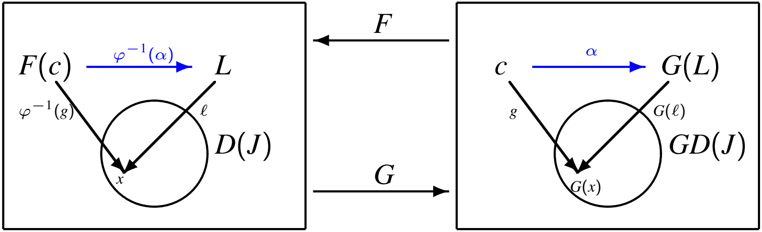
The reverse is true, too.



(3) $\varphi(\theta)$ is unique

The previous argument showed that if $\theta\colon F(c) \to L$ is a cone map, then $\varphi(\theta)\colon c \to G(L)$ is a cone map.

The reverse is true, too.

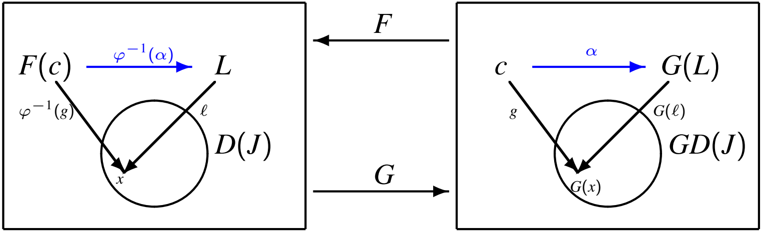


$$\begin{aligned}\varphi_{c,x}(\ell \circ \varphi_{c,L}^{-1}(\alpha)) &= G(\ell \circ \varphi_{c,L}^{-1}(\alpha)) \circ \eta_x = G(\ell) \circ G(\varphi_{c,L}^{-1}(\alpha)) \circ \eta_x \\ &= G(\ell) \circ \varphi_{c,L}(\varphi_{c,L}^{-1}(\alpha)) = G(\ell) \circ \alpha = g.\end{aligned}$$

(3) $\varphi(\theta)$ is unique

The previous argument showed that if $\theta\colon F(c) \to L$ is a cone map, then $\varphi(\theta)\colon c \to G(L)$ is a cone map.

The reverse is true, too.



$$\begin{aligned}\varphi_{c,x}(\ell \circ \varphi_{c,L}^{-1}(\alpha)) &= G(\ell \circ \varphi_{c,L}^{-1}(\alpha)) \circ \eta_x = G(\ell) \circ G(\varphi_{c,L}^{-1}(\alpha)) \circ \eta_x \\ &= G(\ell) \circ \varphi_{c,L}(\varphi_{c,L}^{-1}(\alpha)) = G(\ell) \circ \alpha = g.\end{aligned}$$

Thus cone maps correspond under φ, and the uniqueness of our earlier θ implies the uniqueness of $\varphi(\theta)$.

Consequences and remarks

Consequences and remarks

(1) Direct sum of modules is a colimit, so it is preserved by left adjoints like $\_ \otimes M$.

Consequences and remarks

(1) Direct sum of modules is a colimit, so it is preserved by left adjoints like $\_ \otimes M$.

(2) Directed union is a colimit, so it is preserved by $\_ \otimes M$.

Consequences and remarks

(1) Direct sum of modules is a colimit, so it is preserved by left adjoints like $\_ \otimes M$.

(2) Directed union is a colimit, so it is preserved by $\_ \otimes M$.

(3) The theorem can be used to prove some functors don't have adjoints. E.g., the functor $A \mapsto A \oplus \mathbb{Z}$ has neither a right or left adjoint.

Consequences and remarks

(1) Direct sum of modules is a colimit, so it is preserved by left adjoints like $\_ \otimes M$.

(2) Directed union is a colimit, so it is preserved by $\_ \otimes M$.

(3) The theorem can be used to prove some functors don't have adjoints. E.g., the functor $A \mapsto A \oplus \mathbb{Z}$ has neither a right or left adjoint.

(4) Limits over a fixed category J (when they exist) yield a functor $\varprojlim \colon \mathcal{A}^J \to \mathcal{A}$. Colimits over J yield a functor $\operatorname{colim} = \varinjlim \colon \mathcal{A}^J \to \mathcal{A}$. These are right and left adjoints to the diagonal functor $\Delta \colon \mathcal{A} \to \mathcal{A}^J$. E.g.,

$$\operatorname{Hom}_{\mathcal{A}^J}(\Delta c, D) \cong \operatorname{Hom}_{\mathcal{A}}(c, \varprojlim D).$$

(Lefthand side = cones to D; righthand side = maps to the limit.)

Consequences and remarks

(1) Direct sum of modules is a colimit, so it is preserved by left adjoints like $\_ \otimes M$.

(2) Directed union is a colimit, so it is preserved by $\_ \otimes M$.

(3) The theorem can be used to prove some functors don't have adjoints. E.g., the functor $A \mapsto A \oplus \mathbb{Z}$ has neither a right or left adjoint.

(4) Limits over a fixed category J (when they exist) yield a functor $\varprojlim \colon \mathcal{A}^J \to \mathcal{A}$. Colimits over J yield a functor $\mathrm{colim} = \varinjlim \colon \mathcal{A}^J \to \mathcal{A}$. These are right and left adjoints to the diagonal functor $\Delta \colon \mathcal{A} \to \mathcal{A}^J$. E.g.,

$$\mathrm{Hom}_{\mathcal{A}^J}(\Delta c, D) \cong \mathrm{Hom}_{\mathcal{A}}(c, \varprojlim D).$$

(Lefthand side = cones to D; righthand side = maps to the limit.)

(5) In particular, the limit functor is left exact and the colimit functor is right exact. The situation where such functors are fully exact is important, since homology commutes with exact functors.