Adjoints of Additive Functors

Theorem 1. *If $G\colon \mathcal{A} \to \mathcal{B}$ is an additive functor between preadditive categories and G has a left adjoint, F, then the adjunction bijections*

$$\varphi_{B,A}\colon \mathrm{Hom}_{\mathcal{A}}(F(B), A) \to \mathrm{Hom}_{\mathcal{B}}(B, G(A))$$

are isomorphisms of abelian groups.

Proof. We first explain how $\varphi_{B,A}$ can be computed for all values of A and B if one only knows the values when $A = F(B)$.

The naturality of φ implies that the following diagram commutes whenever $\alpha\colon F(B) \to A$:

$$\begin{tikzcd}
\mathrm{Hom}(F(B), F(B)) \arrow[r, "\varphi_{B,F(B)}"] \arrow[d, "\alpha_*"'] & \mathrm{Hom}(B, GF(B)) \arrow[d, "G(\alpha)_*"'] \\
\mathrm{Hom}(F(B), A) \arrow[r, "\varphi_{B,A}"] & \mathrm{Hom}(B, G(A))
\end{tikzcd}$$

Following the element $\mathrm{id}_{F(B)}$ from the top left homset downward and to the right yields $\varphi_{B,A}(\alpha_*(\mathrm{id}_{F(B)})) = \varphi_{B,A}(\alpha)$. Following it the other way yields $G(\alpha)_*(\varphi_{B,F(B)}(\mathrm{id}_{F(B)})) = G(\alpha) \circ \varphi_{B,F(B)}(\mathrm{id}_{F(B)}))$. So define $\eta_B := \varphi_{B,F(B)}(\mathrm{id}_{F(B)}))$; this is the B-component of a natural transformation $\eta\colon I \to GF$. Our formula is now $\varphi_{B,A}(\alpha) = G(\alpha) \circ \eta_B$ for any $B \in \mathcal{B}$ and any $\alpha \in \mathrm{Hom}(F(B), A)$.

Now we prove that the bijection $\varphi_{B,A}\colon \mathrm{Hom}_{\mathcal{A}}(F(B), A) \to \mathrm{Hom}_{\mathcal{B}}(B, G(A))$ is a group isomorphism. Choose $\alpha, \beta \in \mathrm{Hom}(F(B), A)$. The additivity of G gives that

$$\begin{aligned}
\varphi(\alpha + \beta) &= G(\alpha + \beta) \circ \eta_B \\
&= (G(\alpha) + G(\beta)) \circ \eta_B \\
&= G(\alpha) \circ \eta_B + G(\beta) \circ \eta_B \\
&= \varphi(\alpha) + \varphi(\beta).
\end{aligned}$$

□