Problem 6. (Martinez, Selker). *Given a homomorphism $\varphi : G' \to G$ one can convert any G-module into a G' module by restriction of scalars. Show that such a function induces an additive homomorphism on n-cochains:*

$$\widehat{\varphi} : C^n(G,A) \to C^n(G',A) : f \mapsto f \circ \varphi^n.$$

Show that $\widehat{\varphi}$ maps cocycles to cocycles and coboundaries to coboundaries, hence determines a homomorphism $H^n(G,A) \to H^n(G',A)$.

Solution. Given a group G and a G-module A we use the resolution

$$\cdots \to \mathbb{Z}[G^2] \to \mathbb{Z}[G] \to \mathbb{Z}$$

to compute cohomology as Ext. We can define the maps $\mathbb{Z}[G^{n+1}] \to \mathbb{Z}[G^n]$ on the generators by

$$\begin{aligned}[g_1|\dots|g_n] \mapsto\ & g_1[g_2|\dots|g_n] \\ & + \sum_{i=1}^{n-1}(-1)^i[g_1|\dots|g_ig_{i+1}|\dots|g_n] \\ & + (-1)^n[g_1|\dots|g_{n-1}].\end{aligned}$$

We claim that the map $\widehat{\varphi}$ is an additive homomorphism $C^n(G,A) \to C^n(G',A)$. First note that $\widehat{\varphi}(f) \in C^n(G',A)$, as if $[g_1'|\dots|g_{n-1}'] \in \mathbb{Z}[G'^n]$ then $[\varphi(g_1')|\dots|\varphi(g_{n-1}')] \in \mathbb{Z}[G^n]$. Thus we have that $f \circ \varphi^n : \mathbb{Z}[G'^n] \to A$. Now let $f, h \in C^n(G,A)$ and $g' = [g_1'|\dots|g_{n-1}'] \in \mathbb{Z}[G^n]$. Denote $g_i = \varphi(g_i')$ for $i = 1,\dots,n-1$. Then

$$\begin{aligned}\widehat{\varphi}(f+h)\overline{g'} &= (f+h)[\varphi(g_1')|\dots|\varphi(g_{n-1}')] \\ &= [f(g_1)+h(g_1)|\dots|f(g_{n-1})+h(g_{n-1})] \\ &= [f(g_1)|\dots|f(g_{n-1})] + [h(g_1)|\dots|h(g_{n-1})] \\ &= \widehat{\varphi}(f)(g') + \widehat{\varphi}(h)(g').\end{aligned}$$

Checking that $\widehat{\varphi}$ maps cocycles to cocycles and likewise for coboundaries amounts to checking that $\widehat{\varphi}$ is a chain map of abelian groups, i.e. that $\widehat{\varphi}d^n = d'^n\widehat{\varphi}$, where $d_\bullet$ and $d_\bullet'$ are the boundary maps of $C_\bullet(G,A)$ and $C_\bullet(G',A)$, respectively. Let $f \in C^n(G,A)$ and $g' = [g_1'|\dots|g_n'] \in \mathbb{Z}[G'^{n+1}]$, and again denote $g_i = \varphi(g_i')$ for $i = 1,\dots,n$. Now we compute:

$$\begin{aligned}(f \circ d^n \circ \varphi^n)(g') &= f \circ d^n([g_1|\cdots\dots|g_n]) \\ &= f\left(g_1[g_2|\dots|g_n] + \left(\sum_{i=1}^{n-1}(-1)^i[g_1|\dots|g_ig_{i+1}|\dots|g_n]\right) + (-1)^n[g_1|\dots|g_{n-1}]\right) \\ &= f \circ \varphi^{n-1}\left(g_1'[g_2'|\dots|g_n'] + \left(\sum_{i=1}^{n-1}(-1)^i[g_1'|\dots|g_i'g_{i+1}'|\dots|g_n']\right) + (-1)^n[g_1'|\dots|g_{n-1}']\right) \\ &= \left(f \circ \varphi^{n-1} \circ d'^n\right)(g')\end{aligned}$$

$\square$