Problem 5. *Describe all central extensions of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by $\mathbb{Z}_2$ up to equivalence. Explain why some inequivalent extentions have isomorphic middle factors.*

Solution. Before we start on the problem at hand, we'll recall the catalog of groups of order eight. There are five of them: $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, D_4 (the group of symmetries of the square), and Q_8 (the group of quaterions). Which could feasibly be central extensions of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by $\mathbb{Z}_2$? We need $\mathbb{Z}_2 \times \mathbb{Z}_2$ to inject into the center of the group we put in the middle of the extension. The centers of D_4 and Q_8 are too small to accept a group of order four, and the center of $\mathbb{Z}_8$ is $\mathbb{Z}_8$ itself, which does not have a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, as all subgroups of a cyclic subgroup are cyclic. This leaves just two options, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4 \times \mathbb{Z}_2$. So if we find an element of order more than two in one of our middle groups, it must be isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$.

Also recall that central extensions are those extensions $0 \to K \to G \to Q \to 0$ where the action of the quotient Q on the normal subgroup K is trivial. So once a 2-cocycle f is specified, we can completely describe the group G that goes in the middle: as a set it looks like $(\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{Z}_2$, and its multiplication is given by the following rule. If $k_1, k_2 \in \mathbb{Z}_2 \times \mathbb{Z}_2$ and $q_1, q_2 \in \mathbb{Z}_2$, then $(k_1, q_1)(k_2, q_2) = (k_1 + k_2 + f(q_1, q_2), q_1 q_2)$. (We follow the convention from class that the quotient will be written multiplicatively and the normal subgroup will be written additively. As such, consider the quotient group $\mathbb{Z}_2$ to be the group $\{\pm 1\}$ under multiplication.) We now focus on determining 2-cocycles.

A 2-cocycle is, first of all, a function $(\mathbb{Z}_2)^2 \to \mathbb{Z}_2 \times \mathbb{Z}_2$. If we restrict our attention to normalized 2-cocycles f, then we can fill in for free three of its four values: $f(1,1) = f(1,-1) = f(-1,1) = (0,0)$. The only nontrivial value of f is $f(-1,-1)$, and it can be any element of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Write f_k for 2-cocycle which satisfies $f_k(-1,-1) = k$. So there are four normalized 2-cocycles: $f_{(0,0)}$, $f_{(0,1)}$, $f_{(1,0)}$, and $f_{(1,1)}$. What of 2-coboundaries? The extensions the cocycles create will be distinct only up to coboundaries. In this case, all the coboundaries are zero, so our cocycles are distinct. Work with normalized 2-coboundaries, which arise from normalized 1-cochains. A normalized 1-cochain is a function $g : \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ satisfying $g(1) = 0$. Any choice for $g(-1)$ will then make g into a group homomorphism. Thus $dg(x,y) = g(y) - g(xy) + g(x)$ will always be zero, as claimed.

We've decided that our four distinct 2-cocycles lead to four groups, using the multiplication rule suggested two paragraphs up. To save time, abbreviate $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_{\mathsf{triv}, f_k} \mathbb{Z}_2$ as just G_k. So there are four groups, $G_{(0,0)}$, $G_{(0,1)}$, $G_{(1,0)}$, and $G_{(1,1)}$. These give rise to four extensions; following the same naming convention, let $0 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to G_k \to \mathbb{Z}_2 \to 0$ be denoted E_k.

Each G_k is either isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_2$; which is which? Start with $G_{(0,0)}$. Then $f_{(0,0)}$ is the trivial function that always returns $(0,0)$, so the multiplication in $G_{(0,0)}$ is given by $(k_1, q_1)(k_2, q_2) = (k_1 + k_2, q_1 q_2)$. This is the operation on the direct product, so $G_{(0,0)}$ is the direct product (three cyclic groups of order two). In each of the other groups, $((1,0), -1)$ will have order four. (In fact, any element with -1 in the second coordinate will have order four.) In every group the identity element is $((0,0), 1)$, so if we multiply $((1,0), -1)$ by itself and don't get the identity element, by the remarks above, we must have an element of order four. Check:

$$\begin{aligned}
&\text{in } G_{(1,0)}, \quad ((1,0),-1)\,((1,0),-1) = ((1,0)+(1,0)+(1,0), -1\cdot -1) = ((1,0),1)\\
&\text{in } G_{(0,1)}, \quad ((1,0),-1)\,((1,0),-1) = ((1,0)+(1,0)+(0,1), -1\cdot -1) = ((0,1),1)\\
&\text{in } G_{(1,1)}, \quad ((1,0),-1)\,((1,0),-1) = ((1,0)+(1,0)+(1,1), -1\cdot -1) = ((1,1),1)\,,
\end{aligned}$$

veryifying the claim. Conclude $G_{(1,0)} \cong G_{(0,1)} \cong G_{(1,1)} \cong \mathbb{Z}_4 \times \mathbb{Z}_2$.

We notice that the middle factors of $E_{(0,1)}$ and $E_{(1,0)}$ are both isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$, yet we will see that these two extensions are in fact inequivalent. Suppose that there was some equivalence $(1, \beta, 1)$ between $E_{(0,1)}$ and $E_{(1,0)}$. Then we would get the following commutative diagram:

$$\begin{tikzcd}
\big(E_{(0,1)}\big) & 0 \arrow[r] & \mathbb{Z}_2 \times \mathbb{Z}_2 \arrow[r, "i"] \arrow[d, "1"] \arrow[dr, phantom, "\circlearrowleft"] & G_{(0,1)} \arrow[r, "\epsilon"] \arrow[d, dashed, "\beta"] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z}_2 \arrow[r] \arrow[d, "1"] & 0 \\
\big(E_{(1,0)}\big) & 0 \arrow[r] & \mathbb{Z}_2 \times \mathbb{Z}_2 \arrow[r, "i'"] & G_{(1,0)} \arrow[r, "\epsilon'"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

We see that $i : k \mapsto (k, 1)$ and $i' : k \mapsto (k, 1)$. Then by the commutativity of the diagram $\beta(k, 1) = \beta \circ i(k) = i' \circ 1(k) = (k, 1)$.

Now in $G_{(0,1)}$ we see that there are four elements, $((0,0),-1)$, $((0,1),-1)$, $((1,0),-1)$, and $((1,1),-1)$, of order four, and each of these elements square to $((0,1),1)$. However, the same elements have order four in $G_{(1,0)}$, but they each square to $((1,0),1)$. But $\beta((0,1),1) = ((0,1),1)$, which we see is not the square of an order four element in $G_{(1,0)}$, so such a map β cannot exist, thus $E_{(0,1)}$ and $E_{(1,0)}$ are inequivalent. We can use the same argument to show that even though $E_{(0,1)}$, $E_{(1,0)}$, and $E_{(1,1)}$ all have isomorphic middle factors, the extensions are inequivalent.

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