HOMOLOGICAL ALGEBRA HOMEWORK VI

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Problem 3

Suppose that $G = \{1, g\}$ has two elements, and that A is a G-module.

(a) Show that for $n > 0$

$$H^n(G, A) = \begin{cases} \ker(g+1)/\operatorname{im}(g-1) & n \text{ odd;} \\ \ker(g-1)/\operatorname{im}(g+1) & n \text{ even.} \end{cases}$$

(b) Let $A = \mathbb{C}^*$ and let $G = \{1, g\}$ where g is complex conjugation. Show that $H^n(G, \mathbb{C}^*) = \{0\}$ for even $n > 0$ and $H^n(G, \mathbb{C}^*) \cong \mathbb{Z}_2$ for odd n.

Solution to (a)

We have that $H^n(A, G) \cong \operatorname{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$ where $\mathbb{Z}G$ is the group ring and $\mathbb{Z}$ is the trivial $\mathbb{Z}G$-module. To compute $\operatorname{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$ we begin by finding a projective resolution for the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. In fact we will produce a free resolution. Observe that $\mathbb{Z}G$ is freely generated as a $\mathbb{Z}G$-module by 1 under the action of left multiplication. Then identify the trivial module $\mathbb{Z}$ with the submodule of $\mathbb{Z}G$ generated by the element $g + 1$, which we will denote by $\mathbb{Z}(g + 1) \subset \mathbb{Z}G$, after making the observation that $g+1$ is invariant under left multiplication by 1 and g. Then consider the following free resolution of $\mathbb{Z} \cong \mathbb{Z}(g + 1)$:

$$\cdots \longrightarrow \mathbb{Z}G \xrightarrow{g+1} \mathbb{Z}G \xrightarrow{g-1} \mathbb{Z}G \xrightarrow{g+1} \mathbb{Z}(g+1) \longrightarrow 0.$$

Since we have already observed that $\mathbb{Z}G$ is a free $\mathbb{Z}G$-module, to show this is a projective resolution it remains to define the maps on the generator 1, and check exactness. Each map will be considered as an endomorphism of $\mathbb{Z}G$, because we are considering $\mathbb{Z}(g + 1)$ as a submodule of $\mathbb{Z}G$. The maps $g + 1$ and $g - 1$ are left multiplication by these elements, respectively. We compute:

$$(g + 1)(ng + m1) = (n1 + mg) + (ng + m1) = (n + m)g + (n + m)1.$$

Thus we have $\operatorname{im}(g+1) \subseteq \mathbb{Z}(g+1)$. Furthermore, if $ng+n1 \in \mathbb{Z}(g+1)$, then we have $ng \mapsto ng+n1$, so $\operatorname{im}(g+1) = \mathbb{Z}(g+1)$. Now suppose $(n+m)g+(n+m)1 = 0$. Then $n = -m$, so $\ker(g+1) = \mathbb{Z}(g-1)$. We also compute

$$(g - 1)(ng + m1) = (n1 + mg) - (ng + m1) = (m - n)g + (n - m)1.$$

Thus we have $\operatorname{im}(g-1) \subseteq \mathbb{Z}(g-1)$. Furthermore, if $ng-n1 \in \mathbb{Z}(g-1)$, then we have $ng \mapsto gn-n1$, so $\operatorname{im}(g-1) = \mathbb{Z}(g-1)$. Now suppose $(m-n)g+(n-m)1 = 0$. Then $n = m$, so $\ker(g-1) = \mathbb{Z}(g+1)$. Then we have that $\ker(g + 1) = \operatorname{im}(g - 1)$ and $\ker(g - 1) = \operatorname{im}(g + 1)$, so the above sequence is exact. Then we form the sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \xrightarrow{(g-1)^*} \operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \xrightarrow{(g+1)^*} \operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \longrightarrow \cdots$$

by applying the contravariant functor $\operatorname{Hom}_{\mathbb{Z}G}(\_, A)$ to the deleted resolution. The maps are induced by composition with $g - 1$ or $g + 1$. That is, if $f \in \operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A)$, then

$$(g + 1)^* \colon f \mapsto f \circ (g + 1)$$

and

$$(g - 1)^* \colon f \mapsto f \circ (g - 1).$$

But since f is a $\mathbb{Z}G$-module homomorphism we have

$$(g+1)^*(f)(x) = f((g+1)(x)) = (g+1)f(x)$$

and similarly

$$(g-1)^*(f)(x) = f((g-1)(x)) = (g-1)f(x).$$

Thus $(g+1)^*$ and $(g-1)^*$ are endomorphisms of the $\mathbb{Z}G$-module $\mathrm{Hom}_{\mathbb{Z}G,A}$ defined by left multiplication by the elements $g+1$ and $g-1$. Furthermore, each f is determined by where it maps the generator 1, so the map

$$(1 \mapsto a) \mapsto a$$

gives a $\mathbb{Z}G$-module isomorphism $\mathrm{Hom}_{\mathbb{Z}G,A}(\mathbb{Z}G, A) \cong A$. In which case we may consider the maps $(g+1)^*$ and $(g-1)^*$ as left action by $g+1$ and $g-1$ and our sequence becomes

$$0 \longrightarrow A \xrightarrow{g-1} A \xrightarrow{g+1} A \longrightarrow \cdots$$

Since $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$ is the nth homology of this sequence, we have

$$H^n(G, A) = \begin{cases} \ker(g+1)/\mathrm{im}(g-1) & n \text{ odd;} \\ \ker(g-1)/\mathrm{im}(g+1) & n \text{ even} \end{cases}$$

as required. □

Solution to (b)

Let $A = \mathbb{C}^*$ and let $G = \{1, g\}$ where g is complex conjugation. Let $z = a + bi \in \mathbb{C}^*$. Then either a or b is nonzero. We compute

$$(g+1)(a+bi) = (a-bi)(a+bi) = a^2 + b^2$$

which is the map

$$z \mapsto |z|^2$$

Thus $\mathrm{im}(g+1) = \mathbb{R}_{>0}$ and $\ker(g+1)$ is the unit circle, since $|z|^2 = 1$ iff $|z| = 1$. We also compute

$$(g-1)(a+bi) = \frac{a-bi}{a+bi} = \frac{a^2-b^2}{a^2+b^2} - \frac{2abi}{a^2+b^2}.$$

For z in the kernel of this map we require $2ab = 0$, so either a or b is zero. If $a = 0$ then $(g-1)z = -b^2/b^2 = -1$ and if $b = 0$ then $(g-1)z = a^2/a^2 = 1$. Thus $b = 0$ is both necessary and sufficient for $z \in \ker(g-1)$, hence $\ker(g-1) = \mathbb{R}^*$. The image lies on the unit circle since

$$\left(\frac{a^2-b^2}{a^2+b^2}\right)^2 + \left(\frac{2ab}{a^2+b^2}\right)^2 = \frac{(a^2+b^2)^2}{(a^2+b^2)^2} = 1$$

If z lies on the unit circle to begin with then $g-1$ maps it to

$$(a^2-b^2) - 2abi = (a-bi)^2 = (\overline{z})^2 = z^{-2}$$

since $\overline{z} = 1/z$ for z on the unit circle. Then $z^{-1/2} \mapsto z$, so $g-1$ maps onto the unit circle and we have $\mathrm{im}(g-1)$ is the unit circle. Let $n > 0$. By part (a) we have

$$H^n(G, A) = \begin{cases} \ker(g+1)/\mathrm{im}(g-1) & n \text{ odd;} \\ \ker(g-1)/\mathrm{im}(g+1) & n \text{ even} \end{cases}$$

Let S denote the unit circle. Then we have

$$\ker(g+1)/\mathrm{im}(g-1) = S/S = 0.$$

and also

$$\ker(g-1)/\mathrm{im}(g+1) = \mathbb{R}^*/\mathbb{R}_{>0} \cong \mathbb{Z}_2,$$

since $\{\pm 1\}$ forms a complete set of coset representatives. The result follows. □