ASSIGNMENT 6 - QUESTION 2

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Exercise 2. Calculate $H^1(\mathrm{Aut}(A), A)$ when A is the Klein group.

Proof. Note that $A = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathrm{Aut}(A) = S_3$. This is because A has three non-identity elements all of order two, so they can be permuted in any way to produce an automorphism. We are trying to find the set, $\mathrm{Der}(S_3, A)$, of all derivations of S_3 into A. Once these are found we want to find the principal derivations, $\mathrm{Pder}(S_3, A)$, of S_3 into A. Then we have $H^1(\mathrm{Aut}(A), A) = \mathrm{Der}(S_3, A)/\,\mathrm{Pder}(S_3, A)$. Notice that by derivation we mean normalized 1-cocyle.

It is convenient to work with a presentation of S_3. Let $\sigma \in \mathrm{Aut}(A)$ be defined by

$$\sigma(1, 0) = (1, 1), \text{ and } \sigma(0, 1) = (1, 0),$$

and $\tau \in \mathrm{Aut}(A)$ be defined by

$$\tau(1, 0) = (0, 1) \text{ and } \tau(0, 1) = (1, 0).$$

Notice that $\sigma^3 = \tau^2 = \tau\sigma\tau\sigma = 1$, where $1 \in S_3$ is the identity element. It suffices to define functions on σ and τ and impose the condition that they be derivations. For any derivation $f : \mathrm{Aut}(A) \to A$ we have that $0 = f(\tau^2) = \tau f(\tau) + f(\tau)$. Thus $\tau f(\tau) = f(\tau)$, so $f(\tau)$ is invariant under the action of τ. Hence $f(\tau)$ is either $(0, 0)$ or $(1, 1)$. Notice also that if $f(\sigma) = 0$ then the identity $\tau\sigma = \sigma^2\tau$ yields the following

$$f(\tau) = \sigma^3 f(\tau) = \sigma f(\sigma^2\tau) = \sigma f(\tau\sigma) = \sigma f(\tau).$$

Hence $f(\tau)$ is fixed by the action of σ and so it must be $(0, 0)$. Therefore σ could map to any of four elements and τ could map to any of two elements. So the number of distinct derivations is at most eight. However the restriction that if σ maps to zero then τ maps to zero gives us at most seven distinct derivation of $\mathrm{Aut}(A)$ into A. Recall that the coboundary map $d : A \to \mathrm{Der}(S_3, A)$ is given by

$$d(a, b)(\eta) = \eta(a, b) + (a, b) \text{ for all } (a, b) \in A,\ \eta \in S_3.$$

Let $(a, b) \in \ker d$. Then

$$(0, 0) = d(a, b)(\sigma) = \sigma(a, b) + (a, b) = (a + b, a)$$

This implies that $a = a + b = 0 \in \mathbb{Z}_2$, and hence $(a, b) = (0, 0)$. Therefore d is injective. Thus there are at least four principal derivations in $\mathrm{Der}(S_3, A)$. The number of principal derivations must divide the total number of derivations so there must only be four total derivations, all of which are principal. Thus $H^1(\mathrm{Aut}(A), A) = 0$. $\square$

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