Topics In Algebra, Homework 5

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Problem (5)**.** *Let $p_0, p_1, p_2, ...$ be the sequence of prime numbers.*

a) $t\left(\prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right) = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}$.

b) If D is defined by the short exact sequence

$$0 \to \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n} \to \prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n} \to D \to 0 \tag{1}$$

then there is some extension of D by $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}$ that does not split.

Proof. a) Set theoretically, we can take

$$\prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n} := \left\{ f : \mathbb{N} \to \bigcup_{n\in\mathbb{N}} \mathbb{Z}_{p_n} \ : \ f(n) \in \mathbb{Z}_{p_n} \text{ for all } n \in \mathbb{N} \right\}$$

and

$$\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n} := \left\{ f \in \prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n} \ : \ f(n) \neq 0 \text{ only for finitely many } n \in \mathbb{N} \right\}$$

and it is convenient on this occasion to do so.

To see that $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n} \subseteq t\left(\prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right)$, take $f \in \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}$ different from 0 and let $\{n_0, ..., n_k\} := \{n \in \mathbb{N} \ : \ f(n) \neq 0\}$. Since $\mathbb{Z}_{p_{n_j}}$ is annihilated by $(p_{n_j}) \subset \mathbb{Z}$ for all $j = 0, ..., k$ set $m = p_{n_0} \cdots p_{n_k}$. Clearly then, $mf(n_j) = 0$ for all $j = 0, ...k$. Since $mf(n) = m0 = 0$ for $n \notin \{n_0, ..., n_k\}$ it follows $mf = 0$.

To see that $t\left(\prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right) \subseteq \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}$ take $f \in \prod_{n\in\mathbb{Z}} \mathbb{Z}_{p_n}$ and suppose that $f \notin \oplus_{n\in\mathbb{N}}\mathbb{Z}_{p_n}$. Let m be any non-zero integer and $\{q_0, ..., q_k\}$ be the set of primes dividing m. Since the set $\{p_n \ : \ f(n) \neq 0\}$ is infinite it follows there is some n such that $f(n) \neq 0$ and p_n exceeds each of $q_0, ..., q_k$. Therefore, m does not have p_n as a factor. As $\mathbb{Z}_{p_n}$ is a field and $f(n) \neq 0$ we have $mf(n) \neq 0$ which shows f is not torsion.

b) It is natural to try to compute $\mathrm{Ext}^1(D, \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n})$ but this task will be simplified once we prove that D is a $\mathbb{Q}$ vector space. For this it is sufficient that it be torsion free and divisible. (If you want to check this is sufficient recall that the map $D \to \mathbb{Q} \otimes_{\mathbb{Z}} D$ with $d \mapsto 1 \otimes d$ has kernel $t(D)$ and is surjective if D is divisible). Since (1) is exact we have $D \cong \frac{\prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n}}{\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}} = \frac{\prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n}}{t(\prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n})}$ so D is torsion free. To see D is divisible take any non-torsion $f \in \prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n}$ together with any non-zero integer m. Let $\{q_0, ..., q_k\}$ be the set of primes dividing m. As m has an inverse in $\mathbb{Z}_p$ provided $p \nmid m$, we can define $g \in \prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n}$ by

$$g(n) = \begin{cases} m^{-1}f(n) & \text{if } p_n \nmid m \\ 0 & \text{otherwise} \end{cases}$$

Clearly then $mg(n) = f(n)$ except when $p_n = q_j$ for some $j \in \{0, ..., k\}$ which shows $mg \equiv f \mod \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}$. So D is divisible.

Let κ be the dimension of D as a $\mathbb{Q}$ vector space so that $D \cong \bigoplus_{\alpha\in\kappa} \mathbb{Q}$. Therefore,

$$\operatorname{Ext}^1_{\mathbb{Z}}\left(D, \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right) \cong \operatorname{Ext}^1_{\mathbb{Z}}\left(\bigoplus_{\alpha\in\kappa} \mathbb{Q}, \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right) \cong \prod_{\alpha\in\kappa} \operatorname{Ext}^1_{\mathbb{Z}}\left(\mathbb{Q}, \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right)$$

It is trivial that $\kappa > 0$ because for instance the constant function $f \equiv 1$ is in $\prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n}$ but is not torsion. Therefore, to show that $\operatorname{Ext}^1_{\mathbb{Z}}\left(D, \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right)$ is non-trivial it will be sufficient to find that $\operatorname{Ext}^1_{\mathbb{Z}}\left(\mathbb{Q}, \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right)$ is non-trivial.

Recall that the short exact sequence (1) gives rise to a long exact sequence

$$0 \to \operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Q}, \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right) \to \operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Q}, \prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right) \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, D) \to$$

$$\operatorname{Ext}^1_{\mathbb{Z}}\left(\mathbb{Q}, \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right) \to \operatorname{Ext}^1_{\mathbb{Z}}\left(\mathbb{Q}, \prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right) \to \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, D) \to \cdots$$

From this we are interested in the segment

$$\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Q}, \prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right) \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, D) \to \operatorname{Ext}^1_{\mathbb{Z}}\left(\mathbb{Q}, \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right) \to \operatorname{Ext}^1_{\mathbb{Z}}\left(\mathbb{Q}, \prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right) \tag{2}$$

Now $\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Q}, \prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right) \cong \prod_{n\in\mathbb{N}} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_{p_n})$. Since $\mathbb{Q}$ is divisible every homomorphic image of $\mathbb{Q}$ is divisible. For any prime p, $\mathbb{Z}_p$ only has two subgroups. One can not divide by p in $\mathbb{Z}_p$ so every homomorphic image of $\mathbb{Q}$ in $\mathbb{Z}_p$ must be 0. Thus $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_p)$ is the zero group and we get that $\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Q}, \prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right) = 0$.

Again, we know that $\operatorname{Ext}^1_{\mathbb{Z}}\left(\mathbb{Q}, \prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right) \cong \prod_{n\in\mathbb{N}} \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_{p_n})$ and that for any prime p, $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_p)$ is in one-one correspondence with the extensions of $\mathbb{Q}$ by $\mathbb{Z}_p$. We propose that every short exact sequence of the form

$$0 \to \mathbb{Z}_p \xrightarrow{\iota} A \xrightarrow{\rho} \mathbb{Q} \to 0$$

splits. To establish this proposal we exhibit a section of ρ. Let $\phi : A \to A$ be the group homomorphism $a \mapsto pa$ and put $Q := \operatorname{im}\phi$. Now we claim that $\rho : Q \to \mathbb{Q}$ is an isomorphism. Given $q \in \mathbb{Q}$ take $a \in A$ such that $\rho(a) = q/p$ thus, $\rho(pa) = p\rho(a) = q$ and since $pa \in Q$ it follows $\rho(Q) = \mathbb{Q}$. Now suppose $\rho(a) = 0$ for some $a \in Q$. Let $b \in A$ be an element such that $\phi(b) = a$. Then $\rho(a) = p\rho(b) = 0$ so that $\rho(b) = 0$ and this means $b \in \operatorname{im}\iota$. Now since $p\mathbb{Z}_p = 0$ this means $b \in \ker\phi$. Thus, $\phi(b) = a = 0$ which shows ρ is injective on Q. Therefore, let $s = (\rho\restriction_Q)^{-1}$ and it trivially follows that $\rho \circ s = \mathrm{id}_{\mathbb{Q}}$. Thus, any extension of $\mathbb{Q}$ by $\mathbb{Z}_p$ splits on the right which means it splits. Therefore $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}_p) = 0$, hence $\operatorname{Ext}^1_{\mathbb{Z}}\left(\mathbb{Q}, \prod_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right) = 0$.

So we've shown that (2) is actually

$$0 \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, D) \to \operatorname{Ext}^1_{\mathbb{Z}}\left(\mathbb{Q}, \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right) \to 0$$

which means $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, D) \cong \operatorname{Ext}^1_{\mathbb{Z}}\left(\mathbb{Q}, \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right)$. Since D is a $\kappa > 0$ dimensional $\mathbb{Q}$ vector space, $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, D)$ is non-trivial so that $\operatorname{Ext}^1_{\mathbb{Z}}\left(D, \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right)$ is non-trivial as well. Since every split extension of D by $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}$ corresponds to the zero element of $\operatorname{Ext}^1_{\mathbb{Z}}\left(D, \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}\right)$ it stands to reason that somewhere in the universe there is manifest a non-split extension of D by $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}_{p_n}$.

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