Exercise 5.6

Show that every extension of an abelian group B of bounded exponent by a torsion-free abelian group F splits.

Consider the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q}$. All torsion free abelian groups are flat as $\mathbb{Z}$-modules, so tensoring with F gives another exact sequence: $0 \to \mathbb{Z} \otimes_{\mathbb{Z}} F \to \mathbb{Q} \otimes_{\mathbb{Z}} F$. For any R-module M we know that $M \cong R \otimes_R M$, therefore $F \cong \mathbb{Z} \otimes_{\mathbb{Z}} F$. For brevity, let $V = \mathbb{Q} \otimes_{\mathbb{Z}} F$. Note that V is a $\mathbb{Q}$-vector space, as it is the standard module obtained by extending the scalars of F. We have therefore shown that F is embeddable in V. This embedding is expressible in the following SES: $0 \to F \to V \to V/F \to 0$. Forming the associated LES from the derived functor of $\mathrm{Hom}(\_, B)$ yields an exact sequence $\mathrm{Ext}^1_{\mathbb{Z}}(V, B) \xrightarrow{\varphi} \mathrm{Ext}^1_{\mathbb{Z}}(F, B) \xrightarrow{\psi} 0$. By exercise 4(b), $\mathrm{Ext}^1_{\mathbb{Z}}(V, B)$ is divisible, because all $\mathbb{Q}$ vector spaces are divisible. Similarly, exercise 4(a) shows that $\mathrm{Ext}^1_{\mathbb{Z}}(F, B)$ is annihilated by some positive integer n. However, it is impossible to inject a divisible group into a group of bounded exponent, unless the divisible group is trivial. Because quotients of divisible groups are divisible, this ensures that $\mathrm{im}(\varphi) = 0$. The exactness of $\mathrm{Ext}^1_{\mathbb{Z}}(V, B) \to \mathrm{Ext}^1_{\mathbb{Z}}(F, B) \to 0$ then ensures that $\mathrm{Ext}^1_{\mathbb{Z}}(F, B) = 0$. There is a bijection between ext^1 and Ext^1, in which 0 corresponds to the equivalence class of split extensions. Therefore ext^1 has one element, meaning that all extensions of B by F split.