Assignment V

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5 *Give an example to show that the Universal Coefficient Theorem for Homology may fail for complexes that are not flat.*

Solution. Consider the chain:

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$

with $\mathbb{Q}/\mathbb{Z}$ in the $n-1$-st position of the chain. The module $\mathbb{Q}/\mathbb{Z}$ is not flat: abelian groups are flat iff torsion free. Supposing the Universal Coefficient Theorem holds on $\mathbb{Q}/\mathbb{Z}$, for any module A, the sequence

$$0 \longrightarrow H_n(C) \otimes A \longrightarrow H_n(C \otimes A) \xrightarrow{\varphi} \mathrm{Tor}_1(H_{n-1}(C), A) \longrightarrow 0$$

is exact. However, when we substitute into the sequence, we have

$$0 \longrightarrow 0 \otimes A \longrightarrow H_n(0 \otimes A) \xrightarrow{\varphi} \mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, A) \longrightarrow 0$$

$$0 \longrightarrow 0 \longrightarrow 0 \xrightarrow{\varphi} \mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, A) \longrightarrow 0$$

However, we see that since the sequence is exact, φ is surjective, making $\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, A) = 0$. Since $\mathbb{Q}/\mathbb{Z}$ is not flat, this is not true for every A; this contradiction proves that $\mathbb{Q}/\mathbb{Z}$ is a counterexample to the non-necessity of flatness. (This same argument shows that the Universal Coefficient Theorem will hold for a complex with zero morphisms iff all the objects in the complex are flat.) ∎