Homework 5 Problem 4

Problem 1.

Show that if $mA = \{0\}$, then $m\mathrm{Ext}^1_{\mathbb{Z}}(A, B) = \{0\}$ and that if $nB = \{0\}$, then $n\mathrm{Ext}^1_{\mathbb{Z}}(A, B) = \{0\}$.

Show that if A is uniquely m-divisible (i.e., the map $a \mapsto ma$ is an isomorphism of A onto A), then $\mathrm{Ext}^1_{\mathbb{Z}}(A, B)$ is uniquely m-divisible, and that if B is uniquely n-divisible, then $\mathrm{Ext}^1_{\mathbb{Z}}(A, B)$ is uniquely n-divisible.

◁ **Solution:**

Lemma 0.1. *Suppose that $\lambda_m \colon A \to A, a \mapsto ma$ is multiplication by m. The map*

$$\mathrm{Ext}^1_{\mathbb{Z}}(\lambda_m, B) \colon \mathrm{Ext}^1_{\mathbb{Z}}(A, B) \to \mathrm{Ext}^1_{\mathbb{Z}}(A, B)$$

is also multiplication by m. (As is the map $\mathrm{Ext}^1_{\mathbb{Z}}(A, \lambda_n) \colon \mathrm{Ext}^1_{\mathbb{Z}}(A, B) \to \mathrm{Ext}^1_{\mathbb{Z}}(A, B)$.)

Proof. To determine the behavior of $\mathrm{Ext}^1_{\mathbb{Z}}(\_, B)$ on λ_m first choose and fix any projective resolution of A, say $\ldots \to P_2 \to P_1 \to P_0 \to A$, and look at the following diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & A \arrow[d, "\lambda_m"] \\
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & A
\end{tikzcd}$$

Now an obvious lifting of λ_m is to choose λ_m in each degree which gives rise to.

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, "\lambda_m"] & P_1 \arrow[r, "d_1"] \arrow[d, "\lambda_m"] & P_0 \arrow[r, "d_0"] \arrow[d, "\lambda_m"] & A \arrow[d, "\lambda_m"] \\
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & A
\end{tikzcd}$$

If we apply the contra-variant $\mathrm{Hom}(\cdot, B)$ functor then we get the following

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(A, B) \arrow[r, "d_0^\star"] & \mathrm{Hom}(P_0, B) \arrow[r, "d_1^\star"] & \mathrm{Hom}(P_1, B) \arrow[r] & \cdots \\
0 \arrow[r] & \mathrm{Hom}(A, B) \arrow[r, "d_0^\star"] \arrow[u, "(\lambda_m)^\star"] & \mathrm{Hom}(P_0, B) \arrow[r, "d_1^\star"] \arrow[u, "(\lambda_m)^\star"] & \mathrm{Hom}(P_1, B) \arrow[r] \arrow[u, "(\lambda_m)^\star"] & \cdots
\end{tikzcd}$$

However, $(\lambda_m)^\star$ is actually λ_m since multiplication by m commutes with any abelian group homomorphism. Thus we actually have

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(A,B) \arrow[r, "d_0^\star"] & \mathrm{Hom}(P_0,B) \arrow[r, "d_1^\star"] & \mathrm{Hom}(P_1,B) \arrow[r] & \cdots \\
0 \arrow[r] & \mathrm{Hom}(A,B) \arrow[r, "d_0^\star"] \arrow[u, "(\lambda_m)"] & \mathrm{Hom}(P_0,B) \arrow[r, "d_1^\star"] \arrow[u, "(\lambda_m)"] & \mathrm{Hom}(P_1,B) \arrow[r] \arrow[u, "(\lambda_m)"] & \cdots
\end{tikzcd}$$

Now $\mathrm{ext}^1_{\mathbb{Z}} = \ker(d_1^\star)/Im(d_0^\star)$, and we have ext and Ext are isomorphic. Furthermore, the map on the modules is λ_m and this will induce λ_m on sections. Thus

$$\mathrm{Ext}^1_{\mathbb{Z}}(\lambda_m, B)\colon \mathrm{Ext}^1_{\mathbb{Z}}(A,B) \to \mathrm{Ext}^1_{\mathbb{Z}}(A,B)$$

is multiplication by m.

Now if we dualize the above proof and notice that $(\lambda_m)_\star$ is still λ_m once again because multiplication by m commutes with any abelian group homomorphism then we can get the dual result. □

a) Suppose λ_m is the zero map. Ext is additive and so zero maps are preserved thus

$$\mathrm{Ext}^1_{\mathbb{Z}}(\lambda_m, B)\colon \mathrm{Ext}^1_{\mathbb{Z}}(A,B) \to \mathrm{Ext}^1_{\mathbb{Z}}(A,B)$$

will also be the zero map. However, by the lemma the image under this map is $m\mathrm{Ext}^1_{\mathbb{Z}}(A,B)$ and so $m\mathrm{Ext}^1_{\mathbb{Z}}(A,B) = 0$.

Using the dual part of the lemma and the same argument will yield the desired result for $mB = \{0\}$.

b) Now suppose that the map $a \mapsto ma$ is an isomorphism. Then we apply the lemma. However, $(\lambda_m)^\star$ is now an isomorphism since a functor applied to an isomorphism is still an isomorphism. Thus by the lemma

$$\mathrm{Ext}^1_{\mathbb{Z}}(\lambda_m, B)\colon \mathrm{Ext}^1_{\mathbb{Z}}(A,B) \to \mathrm{Ext}^1_{\mathbb{Z}}(A,B)$$

is the same as multiplication by m and so multiplication by m is an isomorphism. Thus $\mathrm{Ext}^1_{\mathbb{Z}}(A,B)$ is uniquely m divisible.

Using the dual part of the lemma and the same argument will yield the desired result for $\mathrm{Ext}^1_{\mathbb{Z}}(A,B)$ is uniquely n divisible

▷