Problem 3. (Gern, Selker). *Let D be an abelian group. Then D is divisible iff* $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, D) = 0$

Proof. ($\Rightarrow$). Suppose that D is divisible. Then, by Baer's criterion, D is injective so $\operatorname{Ext}^1(\mathbb{Q}/\mathbb{Z}, D) = 0$.

($\Leftarrow$). Suppose that $\operatorname{Ext}^1(\mathbb{Q}/\mathbb{Z}, D) = 0$. Thus also $\operatorname{ext}^1(\mathbb{Q}/\mathbb{Z}, D) = 0$. Consider the exact sequence

$$0 \to \mathbb{Z}_n \xrightarrow{\bar{1} \mapsto \frac{1}{n}} \mathbb{Q}/\mathbb{Z} \xrightarrow{\times n} \mathbb{Q}/\mathbb{Z} \to 0.$$

Resolving and applying $\operatorname{Hom}(\_, D)$ yields the long exact sequence in homology:

$$0 \to \operatorname{Hom}(\mathbb{Q}/\mathbb{Z}, D) \to \cdots \to \operatorname{ext}^1(\mathbb{Q}/\mathbb{Z}, D) \to \operatorname{ext}^1(\mathbb{Z}_n, D) \to \operatorname{ext}^2(\mathbb{Q}/\mathbb{Z}, D) \to \cdots ,$$

Now $\operatorname{ext}^1(\mathbb{Q}/\mathbb{Z}, D) = 0$ by assumption, and $\operatorname{ext}^2(\mathbb{Q}/\mathbb{Z}, D) = 0$ because $\mathbb{Q}/\mathbb{Z}$ and D are abelian groups. Thus we have the exact sequence $0 \to \operatorname{ext}^1(\mathbb{Z}_n, D) \to 0$, whence it follows that $\operatorname{ext}^1(\mathbb{Z}_n, D) \cong \{0\}$. Now $\{0\} = \operatorname{ext}^1(\mathbb{Z}_n, D) \cong D/nD$, so $nD = D$, thus D is divisible. $\square$