HOMOLOGICAL ALGEBRA HOMEWORK V

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Problem 2 Let A and B be R-modules and let A' be a submodule of A. For a homomorphism $\varphi\colon A' \to B$, define the *obstruction* of φ to be $\delta(\varphi)$ where $\delta\colon \mathrm{Hom}(A', B) \to \mathrm{Ext}^1(A/A', B)$ is the connecting homomorphism. Show that $\varphi\colon A' \to B$ can be extended to a homomorphism from A to B iff its obstruction is 0.

Solution We first consider the exact sequence

$$0 \to A' \longrightarrow A \longrightarrow A/A' \to 0$$

obtained from the inclusion of A' into A and the natural map to the quotient. Let

$$0 \to B \longrightarrow Q_0 \longrightarrow Q_1 \longrightarrow Q_2 \to \cdots$$

be an injective resolution of B. The functor $\mathrm{Hom}_R(\_, Q)$ is contravariant and exact when Q is injective. Then it follows that the sequence

$$0 \to \mathrm{Hom}_R(A/A', Q_\bullet) \longrightarrow \mathrm{Hom}_R(A, Q_\bullet) \longrightarrow \mathrm{Hom}_R(A', Q_\bullet) \to 0$$

is a short exact sequence of chain complexes. This induces a long exact sequence on homology groups which, in particular, gives the exact sequence

$$\mathrm{Hom}_R(A, B) \xrightarrow{r} \mathrm{Hom}_R(A', B) \xrightarrow{\delta} \mathrm{Ext}^1(A/A', B)$$

where r is the map induced by the inclusion of A' into A (hence is restriction of morphisms) and δ is the connecting homomorphism. Let $\varphi \in \mathrm{Hom}_R(A', B)$. We have that $\delta(\varphi) = 0$ iff $\varphi \in \ker(\delta) = \mathrm{Im}(r)$. That is to say, iff there exists $\psi \in \mathrm{Hom}_R(A, B)$ such that $\psi\mid_{A'} = r(\psi) = \varphi$. Clearly, such a morphism exists iff φ can be extended to a homomorphism from A to B, as required. $\square$