HOMOLOGICAL ALGEBRA: HOMEWORK 5

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1) Atiyah-MacDonald 7.16: Let A be a Noetherian ring and M a finitely generated A-module. Then the following are equivalent:

(1) M is a flat A-module;
(2) $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module for all primes $\mathfrak{p}$;
(3) $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m}$.

Proof. We begin by proving problem 7.15 from Atiyah-MacDonald:

Lemma 1. *If A is a local Noetherian ring and M is a finitely generated A-module, then the following are equivalent:*

(1) *M is free;*
(2) *M is flat;*
(3) *the mapping $m \otimes M \to A \otimes M$ is injective;*
(4) *$Tor_1^A(k, M) = 0$, where $k = A/\mathfrak{m}$.*

Proof of lemma. (1) $\Rightarrow$ (2): Free modules are flat.

(2) $\Rightarrow$ (3): Consider the short exact sequence $0 \to \mathfrak{m} \to A \to k \to 0$. Since M is flat, tensoring with M produces short exact sequence

$$0 \to \mathfrak{m} \otimes M \to A \otimes M \to k \otimes M \to 0,$$

so $\mathfrak{m} \otimes M \to A \otimes M$ is injective.

(3) $\Rightarrow$ (4): the short exact sequence $0 \to \mathfrak{m} \to A \to k \to 0$ gives the long exact sequence

$$\begin{aligned} \ldots \to \mathrm{Tor}_2^A(k, M) \to \mathrm{Tor}_1^A(m, M) \to \mathrm{Tor}_1^A(A, M) \to \mathrm{Tor}_1^A(k, M)& \\ \to \mathfrak{m} \otimes M \to A \otimes M \to k \otimes M \to 0.& \end{aligned}$$

Since $m \otimes M \to A \otimes M$ is injective,

$$0 = \ker(m \otimes M \to A \otimes M) = \mathrm{im}(\mathrm{Tor}_1^A(k, M) \to m \otimes M).$$

A is a free A-module, so it is projective, so $\ldots \to A \to A \to 0$ is a projective resolution. Thus $\mathrm{Tor}_1^A(A, M) = 0$, and we have that $\mathrm{Tor}_1^A(k, M) \cong \mathrm{im}(\mathrm{Tor}_1^A(k, M) \to m \otimes M) = 0$.

(4) $\Rightarrow$ (1): Consider $M/\mathfrak{m}M$ as an $A/\mathfrak{m}$-vector space. Since M is finitely generated, this vector space is of finite dimension. Fix a basis of $M/\mathfrak{m}M$ and let $x_1, \ldots, x_n \in M$ be representatives of the basis elements. Let N be the submodule of M generated by $x_1, \ldots, x_n$, so that $N + \mathfrak{m}M = M$. Since M is finitely generated and $\mathfrak{m} = J(A)$, by Nakayama's lemma $N = M$. Let F be the free A-module with generators $e_1, \ldots, e_n$ and let $\varphi : F \to M : e_i \mapsto x_i$. If $K = \ker \varphi$, then $0 \to K \to F \xrightarrow{\varphi} M \to 0$, so

$$\ldots \to \mathrm{Tor}_1^A(k, M) \to k \otimes K \to k \otimes F \xrightarrow{1\otimes\varphi} k \otimes M \to 0.$$

Date: April 8, 2010.

is a long exact sequence. Since $\operatorname{Tor}_1^A(k, M) = 0$,

$$0 \to k \otimes K \to k \otimes F \overset{1\otimes\varphi}{\to} k \otimes M \to 0$$

is exact. Now, $k \otimes M$ and $k \otimes F$ are vector spaces of the same dimension, so $1 \otimes \varphi$ is an isomorphism and hence $k \otimes K = 0$. Since $k = A/\mathfrak{m}$, $k \otimes K = 0$ if and only if $\mathfrak{m}K = K$. Since A is Noetherian and K is a submodule of a finitely generated module, K is finitely generated. Thus by Nakayama's lemma again, $K = 0$. Hence $M \cong F$, so M is free. □

We now complete the proof of the problem.

(1) ⇒ (2): Since flatness is a local property, M is flat if and only if $M_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$-module for all primes $\mathfrak{p}$ (Proposition 3.10; Atiyah-MacDonald). Since $A_{\mathfrak{p}}$ is a local Noetherian ring and $M_{\mathfrak{p}}$ is finitely generated, flat modules are free modules. Hence $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module for all primes $\mathfrak{p}$.

(2) ⇒ (3): Maximal ideals are prime.

(3) ⇒ (1): If $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module then $M_{\mathfrak{m}}$ is also a flat $A_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m}$. Since flatness is a local property(Proposition 3.10; Atiyah-MacDonald), this implies that M is a flat A-module. □