# HOMOLOGICAL ALGEBRA HOMEWORK IV

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**Problem** We saw in the last section that if $R = \mathbb{Z}$ (or more generally, if $R$ is a principal ideal domain), a module $B$ is flat iff $B$ is torsionfree. Here is an example of a torsionfree ideal $I$ that is not a flat $R$-module. Let $k$ be a field and set $R = k[x, y]$, $I = (x, y)R$. Show that $k = R/I$ has the projective resolution

$$0 \longrightarrow R \xrightarrow{\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]} R^2 \xrightarrow{(x\ y)} R \longrightarrow k \longrightarrow 0.$$

Then compute that $\mathrm{Tor}_1^R(I, k) \cong \mathrm{Tor}_2^R(k, k) \cong k$, showing that $I$ is not flat.

**Solution** First we give a more explicit definition of the maps involved. The map $R \to k$ is the natural map to the quotient, since we are identifying $k$ with $R/I$. To specify the remaining two maps, note that we are considering $R^2$ as the set of column vectors:

$$R^2 = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} : f, g \in R \right\}.$$

The map $R \to R^2$ is defined by

$$h \mapsto h \begin{bmatrix} -y \\ x \end{bmatrix} = \begin{bmatrix} -yh \\ xh \end{bmatrix}$$

The map $R \to R^2$ is defined by left matrix multiplication by the column vector $(x\ y)$ as follows:

$$\begin{bmatrix} f \\ g \end{bmatrix} \mapsto (x\ y) \begin{bmatrix} f \\ g \end{bmatrix} = xf + yg.$$

To show that the above sequence of maps is a projective resolution, we first show that the sequence is exact. The map $R \to k = R/I$ is the natural map to the quotient, hence surjective. Thus the sequence is exact at $k$. The kernel of the natural map is $I$. Then from the above definition, the image of the map $R^2 \to R$ is the set

$$\{xf + yg : f, g \in R\} = (x, y)R = I$$

so the sequence $R^2 \to R \to k$ is exact. Next consider the kernel of the map $R^2 \to R$. This is the set of column vectors

$$S = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in R^2 : xf - yg = 0 \right\}.$$

Let $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in S$. Then $xf + yg = 0$ so $xf = -yg$. Since $k$ is a field, $k[x, y]$ is a unique factorization domain. The elements $x$ and $y$ are irreducible, so $f = -yh$ and $g = xh'$ for some $h, h' \in R$. Then we have

$$-xyh = -xyh'$$

and hence $h = h'$ by cancellation. Then we have

$$\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} -yh \\ xh \end{bmatrix}$$

Then it follows that the kernel $S$ is contained in the image of the map $R \to R^2$. Now let $\left[\begin{smallmatrix} -yh \\ xh \end{smallmatrix}\right]$ be arbitrary in the image of the map $R \to R^2$. Then we have

$$(x\ y) \begin{bmatrix} -yh \\ xh \end{bmatrix} = xyh - xyh = 0.$$

So the image of the map $R \to R^2$ is contained in the kernel $S$, hence is equal to $S$. Then the sequence $R \to R^2 \to R$ is exact. To show that $0 \to R \to R^2$ is exact, it suffices to show that the map $R \to R^2$ is injective. Let $h, h' \in R$ and suppose that

$$\begin{bmatrix} -yh \\ xh \end{bmatrix} = \begin{bmatrix} -yh' \\ xh' \end{bmatrix}$$

Then in particular $xh = xh'$ and hence $h = h'$ by cancellation. Then the map $R \to R^2$ is injective as required. Thus the sequence

$$0 \to R \to R^2 \to R \to k \to 0$$

is exact. To show that this is a projective resolution of $k$, it remains to show that the $R$-modules $R$ and $R^2$ are projective. But $R$ is isomorphic to the free $R$-module on one generator, and $R^2$ is isomorphic to the free $R$-module on two generators. Free modules are projective, so the above sequence is indeed a projective resolution of $k$. Furthermore, if we consider $I$ as an $R$-module, then the sequence

$$0 \longrightarrow R \xrightarrow{\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]} R^2 \xrightarrow{(x\ y)} I \longrightarrow 0$$

is exact, since $I$ is the image of the map $R^2 \to R$ as we observed earlier. The $R$-modules $\mathrm{Tor}_n(k, k)$ and $\mathrm{Tor}_n(I, k)$ are the $n$th homology modules of the sequences

$$0 \longrightarrow R \otimes_R k \longrightarrow R^2 \otimes_R k \longrightarrow R \otimes_R k \longrightarrow 0$$

and

$$0 \longrightarrow R \otimes_R k \longrightarrow R^2 \otimes_R k \longrightarrow 0$$

respectively, where the maps are the maps from the original sequences tensored with the identity map on $k$. We are interested in $\mathrm{Tor}_2(k, k)$ and $\mathrm{Tor}_1(I, k)$, but in each case this is the homology module computed at $R \otimes_R k$ in the sequence $0 \to R \otimes_R k \longrightarrow R^2 \otimes_R k$, and the maps into and out of $R \otimes_R k$ are the same in both cases. Thus we clearly have $\mathrm{Tor}_2(k, k) \cong \mathrm{Tor}_1(I, k)$. It remains to show that one of these modules is isomorphic to $k$ (as an $R$-module). It suffices to show that $R \otimes_R k \cong k$ and that the map

$$\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right] \otimes \mathrm{id}_k \colon R \otimes_R k \to R^2 \otimes_R k$$

is the zero map. This is because the image of the map from the zero module is $0 \in R \otimes_R k$, so we would have

$$\mathrm{Tor}_2(k, k) \cong \ker\left(\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right] \otimes \mathrm{id}_k\right)/0 \cong R \otimes_R k/0 \cong R \otimes_R k \cong k.$$

We first show that $R \otimes_R k \cong k$. Consider that every element of $R \otimes_R k$ can be expressed as a simple tensor. To see that this is so, it suffices to show the sum of two simple tensors is a simple tensor. Let $r \otimes a, s \otimes b \in R \otimes_R k$. Then we have

$$r \otimes a + s \otimes b = ra \otimes 1 + sb \otimes 1 = (ra + sb) \otimes 1.$$

Let $r \otimes a \in R \otimes_R k$ where $r$ has constant term $b$. Then since $k = R/(x, y)$, we have that

$$r \otimes a = 1 \otimes r \cdot a = 1 \otimes b \cdot a \in 1 \otimes_R k.$$

Then any tensor in $R \otimes k$ can be written as $1 \otimes c$ for some $c \in k$, and from there it is not difficult to check that the map $\phi : R \otimes k \to k : 1 \otimes c \mapsto c$ is an $R$-module isomorphism. Thus $R \otimes k \cong k$ as desired.

It remains to show that the map $\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right] \otimes \mathrm{id}_k \colon R \otimes_R k \to R^2 \otimes_R k$ is the zero map. Let $1 \otimes c \in R \otimes_R k$. We see that

$$\begin{aligned}
\left(\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right] \otimes \mathrm{id}_k\right)(1 \otimes c) &= \left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right] \otimes c \\
&= \left[\begin{smallmatrix} -y \\ 0 \end{smallmatrix}\right] \otimes c + \left[\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right] \otimes c \\
&= \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \otimes -y \cdot c + \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \otimes x \cdot c \\
&= \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \otimes 0 + \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \otimes 0 \\
&= 0,
\end{aligned}$$

so $\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right] \otimes \mathrm{id}$ is the zero map. The result follows. $\square$