HOMOLOGICAL ALGEBRA
ASSIGNMENT 4

CHRIESTENSON, LI

Problem 6: (Weibel Exercise 3.1.3) Show that $\operatorname{Tor}_1^R(R/I, R/J) \cong \frac{I \cap J}{IJ}$ for any right ideal I and left ideal J in the ring R.

Proof: We first note that from the short exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

We get the derived long exact sequence by tensoring on the right with R/J

$$\cdots \longrightarrow \operatorname{Tor}_1^R(R/I, R/J) \longrightarrow I \otimes_R R/J \longrightarrow R \otimes_R R/J \longrightarrow R/I \otimes_R R/J \longrightarrow 0$$

Now notice that $R \otimes_R R/J \cong R/J$. We also may define maps $\eta : I/IJ \to I \otimes_R R/J$ and $\nu : I \otimes_R R/J \to I/IJ$ by $\eta(i + IJ) = 1 \otimes (i + J)$, and $\nu(j \otimes (r + J)) = jr + IJ$, for any $i, j \in I$ and $r \in R$. It is easy to see that η is a well defined R-module homomorphism since $IJ \subseteq J$, and ν is clearly a well defined R-module homomorphism by the universal property of tensor products. We also have that

$$\eta\nu(j \otimes (r + J)) = \eta(jr + IJ) = 1 \otimes (jr + J) = j \otimes (r + J)$$

and

$$\nu\eta(i + IJ) = \nu(1 \otimes (i + J)) = i + IJ$$

Thus we have that $I/IJ \cong I \otimes_R R/J$.

Now using the short exact sequences with the obvious inclusion and projection maps

$$0 \longrightarrow IJ \longrightarrow I \longrightarrow I/IJ \longrightarrow 0$$

and

$$0 \longrightarrow J \longrightarrow R \longrightarrow R/J \longrightarrow 0$$

and the above isomorphisms we have the following diagram with the middle two rows exact

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & \mathrm{Tor}_1^R\left(\frac{R}{I},\frac{R}{J}\right) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & IJ & \longrightarrow & I & \longrightarrow & I\otimes_R \frac{R}{J} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J & \longrightarrow & R & \longrightarrow & R\otimes_R \frac{R}{J} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \frac{J}{IJ} & & \frac{R}{I} & & \frac{R}{I}\otimes_R \frac{R}{J} & &
\end{array}$$

where the left two vertical maps in the middle are inclusion maps. Notice the right column is exact by what was said at the beginning, and the other two columns are made exact by inserting the appropriate cokernels with their unique morphisms at the bottom.

It is easy to see that everything in the above diagram commutes except maybe the right square

$$\begin{array}{ccc}
I & \xrightarrow{\eta p_1} & I\otimes_R R/J\\
{\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle j\otimes 1_{R/J}}\\
R & \xrightarrow{tp_2} & R\otimes_R R/J
\end{array}$$

Where the p_i's are the natural projections, j is the natural inclusion and t is the isomorphism $R/J \cong R\otimes_R R/J : r+J \mapsto 1\otimes(r+J)$. To see this is commutative we let $i\in I$ then

$$(j\otimes 1_{R/J})\eta p_1(i) = (j\otimes 1_{R/J})\eta(i+IJ) = 1\otimes(i+J) = 1\otimes p_2(i) = tp_2(i) = tp_2 j(i).$$

Thus the above diagram is commutative.

Since the middle two rows are exact, the snake lemma assures us that there is a long exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^R(R/I,R/J) \longrightarrow J/IJ \longrightarrow R/I \longrightarrow R/I\otimes_R R/J \longrightarrow 0$$

Thus we have that $\mathrm{Tor}_1^R(R/I,R/J)$ is isomorphic to the kernel of the induced map $\iota : J/IJ \to R/I$. Since ι is induced from the inclusion $J\to R$ we have that ι is given by $\iota(j+IJ) = j+I$ for any $j\in J$. It is clear that if $j\in J$ is such that $\iota(j+IJ)=0+I$ then $j\in I$, hence $I\cap J/IJ \subseteq \ker(\iota)$. Conversely if $j\in J$ such that $j+I\notin I\cap J/IJ$ then we have that $j\notin I$ so $\iota(j)\neq 0+J$. Thus $\ker(\iota)\subseteq I\cap J/IJ$. Therefore we have that $\mathrm{Tor}_1^R(R/I,R/J)\cong \ker(\iota) = I\cap J/IJ$. This is what we wanted to prove.