Exercise 4.5

A short exact sequence $0 \to A \to B \to C \to 0$ is pure exact if it remains exact under tensoring with an arbitrary module. Show that C is flat iff any short exact sequence ending in C is pure exact.

Suppose that C is flat, let $0 \to A \to B \to C \to 0$ be a SES ending in C, and let X be an arbitrary module. Then $\mathrm{Tor}_1(X, C) = 0$, hence the long exact sequence reads

$$\mathrm{Tor}_1(X,A) \longrightarrow \mathrm{Tor}_1(X,B) \longrightarrow 0 \longrightarrow X \otimes A \longrightarrow X \otimes B \longrightarrow X \otimes C \longrightarrow 0$$

Therefore tensoring with X preserves the exactness of $0 \to A \to B \to C \to 0$.

Suppose now that any SES ending in C is pure exact. Every module is the quotient of some free module. Let this quotient be expressed by the SES $0 \to A \to B \to C \to 0$, where B is free. Free modules are flat, therefore $\mathrm{Tor}_1(X, B) = 0$, and the associated long exact sequence is therefore

$$\mathrm{Tor}_1(X,A) \longrightarrow 0 \longrightarrow \mathrm{Tor}_1(X,C) \xrightarrow{\delta} X \otimes A \longrightarrow X \otimes B \longrightarrow X \otimes C \longrightarrow 0$$

By assumption, $0 \to A \to B \to C \to 0$ is pure, hence $\mathrm{im}(\delta) = 0$. Exactness at $\mathrm{Tor}_1(X, C)$ implies that δ is injective, therefore $\mathrm{Tor}_1(X, C) = 0$. This implies that C is flat, as desired.